THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The one hundred and twentieth regular meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York City on Saturday, October 29, 1904, extending through the usual morning and afternoon sessions. The following twenty-four members were present:

Professor E. W. Brown, Professor F. N. Cole, Dr. W. S. Dennett, Dr. William Findlay, Professor T. S. Fiske, Dr. A. S. Gale, Mr. S. A. Joffe, Dr. Edward Kasner, Dr. G. H. Ling, Mr. L. L. Locke, Dr. Max Mason, Professor W. H. Metzler, Professor James Pierpont, Miss Amy Rayson, Dr. I. E. Rabinovitch, Professor Charlotte A. Scott, Professor D. E. Smith, Mr. E. R. Smith, Professor Henry Taber, Professor H. D. Thompson, Professor E. B. Van Vleck, Professor J. M. Van Vleck, Mr. H. E. Webb, Miss E. C. Williams.

The President of the Society, Professor Thomas S. Fiske, occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. R. P. Baker, Union Academy, Anna, Ill.; Dr. W. H. Bussey, Evanston, Ill.; Mr. A. M. Curtis, State Normal School, Oneonta, N. Y.; Professor G. R. Dean, University of Missouri; Professor R. R. Fleet, William Jewell College, Liberty, Mo.; Professor E. D. Grant, Michigan College of Mines; Mr. J. E. Higdon, Shortridge High School, Indianapolis, Ind.; Dr. L. C. Karpinski, University of Michigan; Dr. O. C. Lester, Yale University; Mr. Arthur Ranum, University of Wisconsin; Mr. C. H. Sisam, U. S. Naval Academy; Miss Adelaide Smith, Huguenot College, Wellington, Cape Colony; Professor C. M. Snelling, University of Georgia; Professor Eduard Study, University of Bonn. Five applications for membership in the Society were received.

A list of nominations for officers and other members of the Council was adopted and ordered placed on the official ballot for the annual election. A committee consisting of Professor Legras and Dr. Kasner was appointed to audit the Treasurer's accounts for the year 1904.

The following papers were presented at this meeting:

(1) Dr. EDWARD KASNER: "Contact transformations and related systems of curves."

(2) Dr. MAX MASON: "The doubly periodic solutions of $\Delta u + \lambda A(x, y)u = f(x, y)$."

(3) Professor E. B. VAN VLECK: "A proof of some theorems on pointwise discontinuous functions."

(4) Professor HENRY TABER: "Hypercomplex number systems."

(5) Mr. J. C. MOREHEAD: "Note on a theorem of Lucas on Fermat's numbers."

(6) Professor E. W. BROWN: "On the completion of the new lunar theory."

(7) Professor VIRGIL SNYDER: "Quintic scrolls having a tacnodal or an oscnodal conic."

(8) Professor G. A. MILLER: "Groups of subtraction and division."

Mr. Morehead's paper was communicated to the Society through Professor Pierpont. In the absence of the authors, Mr. Morehead's paper was read by Professor Pierpont, and the papers of Professor Snyder and Professor Miller were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. A contact transformation of the plane converts the ∞^2 points into ∞^2 curves C_1, and at the same time converts ∞^2 curves C into points. Two doubly infinite systems of curves C and C_1 which can be obtained in this way are termed *related* in Dr. Kasner's paper. They may be represented by a single equation $\Omega(x, y; x_1, y_1) = 0$, involving two sets of point coördinates, the system C arising when x_1, y_1 are regarded as parameters and the system C_1 when x, y are the parameters. All the systems related to a given system are equivalent under the group of all point transformations. When a system admits a group of point transformations the same is true of any related system. The general discussion is supplemented by a detailed discussion of related systems of circles, especially of systems related to themselves.

2. Picard has studied the doubly periodic solutions of

$$\Delta u - u = 0 \tag{1}$$

which possess logarithmic singularities. No periodic solution, except zero, of this equation exists which is everywhere finite. In Dr. Mason's paper the existence of everywhere finite doubly periodic solutions of

$$\Delta u + \lambda A(x, y)u = f(x, y) \tag{2}$$

is considered, where λ is a parameter, A and f are doubly periodic. A doubly periodic solution of (1), given by Picard, is used as a Green's function to reduce the differential equation (2) to a functional equation of a type studied by Fredholm. Theorems are deducted from Fredholm's results, and from the consideration of equation (2), with right member zero, as the necessary condition of an isoperimetric calculus of variations problem. There exists an infinite series of special values of λ for which the homogeneous equation possesses finite doubly periodic solutions. These solutions are the solutions of the minimum problems. The equation (2) possesses a unique doubly periodic solution for any value of λ not a special value, and has a doubly periodic solution for a special value of λ when and only when $f(x, y)$ satisfies n integral conditions, where n is the number of linearly independent solutions of the homogeneous equation for this value of λ. The paper will appear in *Liouville's Journal*.

3. The paper of Professor Van Vleck contained a proof of some six or seven theorems on pointwise discontinuous functions, due mostly to Baire (*Annali di Matematica*, 1899). The demonstration which had been previously given by Baire was based upon the concept of "upper continuity." The introduction of this concept is, however, unnecessary, and the proof was reduced by Professor Van Vleck to yet simpler and more fundamental principles. It was also shown that in one of the theorems more restriction had been imposed upon the function involved than was necessary to establish the conclusion.

4. Professor Taber's paper showed how to reduce to the determination of certain nilpotent number systems the problem of the enumeration of all types of number systems in n variables (for an arbitrary value of n), into one or other of which any given number systems in n variables can be transformed by a transformation rational in the domain of the constants of

multiplication of the given system. It contained also generalizations with respect to a restricted domain of rationality of Cartan's theorems given in the *Comptes Rendus* for 1897. This paper appeared in part in the *Transactions* for October.

5. Mr. Morehead pointed out a singular error in the theorem announced by Lucas on page XII of the preface of his Théorie des nombres: In order that the number $2^{2^n}+1$ be prime, it is necessary and sufficient that it should be a divisor of the number $3^{2^{2^n}}+1$. This statement which is quoted without correction in Pascal's Repertorio, volume 1, page 551, and again in the German edition of the latter work, volume 1, page 516, is obviously at odds with the Fermat law $3^{p-1} \equiv +1 \pmod{p}$.

6. Professor Brown's paper consisted of a general account of the work required to solve that part of the lunar problem which depends on the sun's action. The solution has been under way for the last twelve years and it has just been completed.

7. In volume 9 of the BULLETIN, Professor Snyder discussed five types of quintic scrolls having three double conics. The present paper shows that two or all three of the double conics may become consecutive, and derives the equation of each form. When two conics approach coincidence, the two generators through each point of one of them must lie in a plane containing the tangent to the conic at that point. The point correspondence on the conic is discussed and compared with the Noether-Wiman depiction of the surface on a twisted cubic curve. Two types exist with a tacnodal conic, and one with an osculating conic. The paper will appear in full in the BULLETIN.

8. Let x_1, x_2 represent any two fixed numbers, while n represents an arbitrary number. The two operations $x_1 - n, x_2/n$ generate a group of finite order only when x_1 and x_2 satisfy certain conditions. In the four cases when $x_1 = 0$, $x_1^2 = x_2$, $x_1^2 = 2x_2$, or $x_1^2 = 3x_2$ the corresponding groups are respectively of orders 4, 6, 8, 12. All of these groups are of the dihedral rotation type and the two operations $x_1 - n$, x_2/n cannot generate any other group which transforms every rational point of the plane into a rational point. Hence these four groups are of fundamental importance in the theory of point transforma-

tions of the plane. Every dihedral rotation group can be represented as a subtraction and division group, and the direct interpretation of these operations is always a dihedral rotation group. The operations may be resolved, when complex numbers are employed, so as to lead to other interesting groups.

F. N. COLE,
Secretary.

THE FUNDAMENTAL CONCEPTIONS AND METHODS OF MATHEMATICS.

ADDRESS DELIVERED BEFORE THE DEPARTMENT OF MATHEMATICS OF THE INTERNATIONAL CONGRESS OF ARTS AND SCIENCE, ST. LOUIS, SEPTEMBER 20, 1904.

BY PROFESSOR MAXIME BÔCHER.

I. *Old and New Definitions of Mathematics.*

I am going to ask you to spend a few minutes with me in considering the question: what is mathematics? In doing this I do not propose to lay down dogmatically a precise definition; but rather, after having pointed out the inadequacy of traditional views, to determine what characteristics are common to the most varied parts of mathematics but are not shared by other sciences, and to show how this opens the way to two or three definitions of mathematics, any one of which is fairly satisfactory. Although this is, after all, merely a discussion of the meaning to be attached to a name, I do not think that it is unfruitful, since its aim is to bring unity into the fundamental conceptions of the science with which we are concerned. If any of you, however, should regard such a discussion of the meaning of words as devoid of any deeper significance, I will ask you to regard this question as merely a bond by means of which I have found it convenient to unite what I have to say on the fundamental conceptions and methods of what, with or without definition, we all of us agree to call mathematics.

The old idea that mathematics is the science of quantity, or that it is the science of space and number, or indeed that it can be characterized by any enumeration of several more or less heterogeneous objects of study, has pretty well passed away among those mathematicians who have given any thought to

the question of what mathematics really is. Such definitions, which might have been intelligently defended at the beginning of the nineteenth century, became obviously inadequate as subjects like projective geometry, the algebra of logic, and the theory of abstract groups were developed; for none of these has any necessary relation to quantity (at least in any ordinary understanding of that word), and the last two have no relation to space. It is true that such examples have had little effect on the more or less orthodox followers of Kant, who regard mathematics as concerned with those conceptions which are obtained by direct intuition of time and space without the aid of empirical observation. This view seems to have been held by such eminent mathematicians as Hamilton and DeMorgan; and it is a very difficult position to refute, resting as it does on a purely metaphysical foundation which regards it as certain that we can evolve out of our inner consciousness the properties of time and space. According to this view the idea of quantity is to be deduced from these intuitions; but one of the facts most vividly brought home to pure mathematicians during the last half century is the fatal weakness of intuition when taken as the logical source of our knowledge of number and quantity.*

The objects of mathematical study, even when we confine our attention to what is ordinarily regarded as *pure* mathematics are, then, of the most varied description; so that, in order to reach a satisfactory conclusion as to what really characterizes mathematics, one of two methods is open to us. On the one hand we may seek some hidden resemblance in the various objects of mathematical investigation, and having found an aspect common to them all we may fix on this as the one true object of mathematical study. Or, on the other hand, we may abandon the attempt to characterize mathematics by means of its *objects of study*, and seek in its *methods* its distinguishing characteristic. Finally there is the possibility of our combining these two points of view. The first of these methods is that of Kempe, the second will lead us to the definition of Benjamin Peirce, while the third has recently been elaborated at great length by Russell. Other mathematicians have naturally followed out more or less consistently the same ideas, but I shall nevertheless take the liberty of using the names

* I refer here to such facts as that there exist continuous functions without derivatives, whereas the direct untutored intuition of space would lead anyone to believe that every continuous curve has tangents.

Kempe, Peirce, and Russell as convenient designations for these three points of view. These different methods of approaching the question lead finally to results which, without being identical, still stand in the most intimate relation to one another, as we shall now see. Let us begin with the second method.

II. *Peirce's Definition.*

More than a third of a century ago Benjamin Peirce wrote:* *Mathematics is the science which draws necessary conclusions.* According to this view there is a mathematical element involved in every enquiry in which exact reasoning is used. Thus, for instance,† a jury listening to the attempt of the counsel for the prisoner to prove an alibi in a criminal case might reason as follows. "If the witnesses are telling the truth when they say that the prisoner was in St. Louis at the moment the crime was committed in Chicago, and if it is true that a person cannot be in two places at the same time, it follows that the prisoner was not in Chicago when the crime was committed." This, according to Peirce, is a bit of mathematics; while the further reasoning by which the jury would decide whether or not to believe the witnesses, and the reasoning (if they thought any necessary) by which they would satisfy themselves that a person cannot be in two places at once, would be inductive reasoning which can give merely a high degree of probability to the conclusion, but never certainty. This mathematical element may be, as the example just given shows, so slight as not to be worth noticing from a practical point of view. This is almost always the case in the transactions of daily life and in the observational sciences. If, however, we turn to such subjects as chemistry and mineralogy, we find the mathematical element of considerable importance, though still subordinate. In physics and astronomy its importance is much greater. Finally in geometry, to mention only one other science, the mathematical element predominates to such an extent that this science has been commonly rated a branch of pure mathematics, whereas, according to Peirce, it is as much a branch of applied mathematics as is, for instance, mathematical physics.

It is clear from what has just been said that, from Peirce's

* Linear associative algebra. Lithographed 1870. Reprinted in the *American Journal of Mathematics*, vol. 4.

† This illustration was suggested by the remarks by J. Richard: Sur la philosophie des mathématiques, Paris, Gauthier-Villars, 1903, p. 50.

point of view, mathematics does not necessarily concern itself with quantitative relations, and that any subject becomes capable of mathematical treatment as soon as it has secured data from which important consequences can be drawn by exact reasoning. Thus, for example, even though psychologists be right when they assure us that sensations and the other objects with which they have to deal cannot be measured, we need still not necessarily despair of one day seeing a mathematical psychology, just as we already have a mathematical logic.

I have said enough, I think, to show what relation Peirce's conception of mathematics has to the applications. Let us then turn to the definition itself and examine it a little more closely. You have doubtless already noticed that the phrase: "the science which draws necessary conclusions" contains a word which is very much in need of elucidation. What is a *necessary* conclusion? Some of you will perhaps think that the conception here involved is one about which, in a concrete case at least, there can be no practical diversity of opinion among men with well trained minds; and in fact when I spoke a few minutes ago about the reasoning of the jurymen when listening to the lawyer trying to prove an alibi, I assumed tacitly that this is so. If this really were the case, no further discussion would be necessary, for it is not my purpose to enter into any purely philosophical speculations. But unfortunately we cannot dismiss the matter in this way; for it has happened not infrequently that the most eminent men, including mathematicians, have differed as to whether a given piece of reasoning was exact or not; and, what is worse, modes of reasoning which seem absolutely conclusive to one generation no longer satisfy the next, as is shown by the way in which the greatest mathematicians of the eighteenth century used geometric intuition as a means of drawing what they regarded as necessary conclusions.*

I do not wish here to raise the question whether there is such a thing as absolute logical rigor, or whether this whole concep-

* All writers on elementary geometry from Euclid down almost to the close of the nineteenth century use intuition freely, though usually unconsciously, in obtaining results which they are unable to deduce from their axioms. The first few demonstrations of Euclid are criticised from this point of view by Russell in his Principles of Mathematics, vol. 1, p. 404–407.—Gauss's first proof (1799) that every algebraic equation has a root gives a striking example of the use of intuition in what was intended as an absolutely rigorous proof by one of the greatest and at the same time most critical mathematical minds the world has ever seen.

tion of logical rigor is a purely psychological one bound to change with changes in the human mind. I content myself with expressing the belief, which I will try to justify a little more fully in a moment, that as we never have found an immutable standard of logical rigor in the past, so we are not likely to find it in the future. However this may be, so much we can say with tolerable confidence, as past experience shows, that no reasoning which claims to be exact can make any use of intuition, but that it must proceed from definitely and completely stated premises according to certain principles of formal logic. It is right here that modern mathematicians break sharply with the tradition of *a priori* synthetic judgments (that is, conclusions drawn from intuition) which, according to Kant, form an essential part of mathematical reasoning.

If then we agree that "necessary conclusions" must, in the present state of human knowledge, mean conclusions drawn according to certain logical principles from definitely and completely stated premises, we must face the question as to what these principles shall be. Here, fortunately, the mathematical logicians from Boole down to C. S. Peirce, Schröder, and Peano have prepared the field so well that of late years Peano and his followers * have been able to make a rather short list of logical conceptions and principles upon which it would seem that all exact reasoning depends.† We must remember, however, when we are tempted to put implicit confidence in certain fundamental logical principles, that, owing to their extreme generality and abstractness, no very great weight can be attached to the mere fact that these principles appeal to us as obviously true; for, as I have said, other modes of reasoning which are now universally recognized as faulty have appealed in just this way to the greatest minds of the past. Such confidence as we feel must, I think, come from the fact that those modes of reasoning which we trust have withstood the test of use in an immense number of cases and in very many fields. This is the severest test to which any theory can be put, and if it does not break down under it we may feel the greatest confidence that, at least in cognate fields, it will prove serviceable. But we can never be sure. The accepted modes of exact reasoning may any day lead to a contradiction which would show that

* And, independently, Frege.

† It is not intended to assert that a single list has been fixed upon. Different writers naturally use different lists.

what we regard as universally applicable principles are in reality applicable only under certain restrictions. *

To show that the danger which I here point out is not a purely fanciful one, it is sufficient to refer to a very recent example. Independently of one another Frege and Russell have built up the theory of arithmetic from its logical foundations. Each starts with a definite list of apparently self-evident logical principles, and builds up a seemingly flawless theory. Then Russell discovers that his logical principles when applied to a very general kind of logical *class* lead to an absurdity; and both Frege and Russell have to admit that something is wrong with the foundations which looked so secure. Now there is no doubt that these logical foundations will be somehow recast to meet this difficulty, and that they will then be stronger than ever before.† But who shall say that the same thing will not happen again?

It is commonly considered that mathematics owes its certainty to its reliance on the immutable principles of formal logic. This, as we have seen, is only half the truth imperfectly expressed. The other half would be that the principles of formal logic owe such degree of permanence as they have largely to the fact that they have been tempered by long and varied use by mathematicians. "A vicious circle!" you will perhaps say. I should rather describe it as an example of the process known to mathematicians as the method of successive approximations. Let us hope that in this case it is really a convergent process, as it has every appearance of being.

But to return to Peirce's definition. From what are these necessary conclusions to be drawn? The answer clearly implied is: from any premises sufficiently precise to make it possible to draw necessary conclusions from them. In geometry, for instance, we have a large number of intuitions and fixed

* If the view which I here maintain is correct it follows that if the term 'absolute logical rigor' has a meaning, and if we should some time arrive at this absolute standard, the only indication we should ever have of the fact would be that for a long period, several thousand years let us say, the logical principles in question had stood the test of use. But this state of affairs might equally well mean that during that time the human mind had degenerated, at least with regard to some of its functions. Consider for instance the twenty centuries following Euclid when, without doubt, the high tide of exact thinking attained during Euclid's generation had receded.

† Cf. Poincaré's view in La science et l'hypothèse, p. 179, according to which a theory never renders a greater service to science than when it breaks down.

beliefs concerning the nature of space: it is homogeneous and isotropic, infinite in extent in every direction, etc.; but none of these ideas, however clearly defined they may at first sight seem to be, gives any hold for exact reasoning. This was clearly perceived by Euclid, who therefore proceeded to lay down a list of axioms and postulates, that is specific facts which he assumes to be true, and from which it was his object to deduce all geometric propositions. That his success here was not complete is now well known, for he frequently assumes unconsciously further data which he derives from intuition; but his attempt was a monumental one.

III. *The Abstract Nature of Mathematics.*

Now a further self-evident point, but one to which attention seems to have been drawn only during the last few years, is this: since we are to make no use of intuition, but only of a certain number of explicitly stated premises, it is not necessary that we should have any idea what the nature of the objects and relations involved in these premises is.* I will try to make this clear by a simple example. In plane geometry we have to consider, among other things, points and straight lines. A point may have a peculiar relation to a straight line which we express by the words: the point lies on the line. Now one of the fundamental facts of plane geometry is that two points determine a line, that is if two points are given, there exists one and only one line on which both points lie. All the facts that I have just stated correspond to clear intuitions. Let us, however, eliminate our intuition of what is meant by a point, a line, a point lying on a line. A slight change of language will make it easy for us to do this. Instead of points and lines, let us speak of two different kinds of objects, say A-objects and B-objects; and instead of saying that a point lies on a line we will simply say that an A-object bears a certain relation R to a B-object. Then the fact that two points determine a line will be expressed by saying: If any two A-objects are given, there exists one and only one B-object to which they both bear the relation R. This statement, while it does not force on us any specific intuitions, will serve as a basis for mathematical reason-

* This was essentially Kempe's point of view in the papers to be referred to presently. In the geometric example which follows it was clearly brought out by H. Wiener: *Jahresbericht d. deutschen Mathematiker-Vereinigung*, vol. 1 (1891), p. 45.

ing * just as well as the more familiar statement where the terms *points* and *lines* are used. But more than this. Our A-objects, our B-objects and our relation R may be given an interpretation, if we choose, very different from that we had at first intended.

We may, for instance, regard the A-objects as the straight lines in a plane, the B-objects as the points in the same plane (either finite or at infinity), and when an A-object stands in the relation R to a B-object this may be taken to mean that the line passes through the point. Our statement would then become: Any two lines being given, there exists one and only one point through which they both pass. Or we may regard the A-objects as the men in a certain community, the B-objects as the women, and the relation of an A-object to a B-object as friendship. Then our statement would be: In this community any two men have one, and only one, woman friend in common.

These examples are, I think, sufficient to show what is meant when I say that we are not concerned in mathematics with the nature of the objects and relations involved in our premises, except in so far as their nature is exhibited in the premises themselves. Accordingly mathematicians of a critical turn of mind, during the last few years, have adopted more and more a purely nominalistic attitude towards the objects and relations involved in mathematical investigation. This is, of course, not the crude mixture of nominalism and empiricism of the philosopher Hobbes, whose claim to mathematical fame, it may be said in passing, is that of a circle squarer.† The nominalism of the present day mathematician consists in treating the objects of his investigation and the relations between them as mere symbols. He then states his propositions, in effect, in the following form: If there exist any objects in the physical or mental world with relations among themselves which satisfy the conditions which I have laid down for my symbols, then such and such facts will be true concerning them.

It will be seen that, according to Peirce's view, the mathematician *as such* is in no wise concerned with the source of his premises or with their harmony or lack of harmony with any part of the external world. He does not even assert that any

* In conjunction of course with further postulates with which we need not here concern ourselves.

† Hobbes practically obtains as the ratio of a circumference to its diameter the value $\sqrt{10}$. Cf. for instance Molesworth's edition of Hobbes's English Works, vol. 7, p. 431.

objects really exist which correspond to his symbols. Mathematics may therefore be truly said to be the most abstract of all sciences, since it does not deal directly with reality.*

This, then, is Peirce's definition of mathematics. Its advantages in the direction of unifying our conception of mathematics and of assigning to it a definite place among the other sciences are clear. What are its disadvantages? I can see only two. First that, as has been already remarked, the idea of drawing necessary conclusions is a slightly vague and shifting one. Secondly that it lays exclusive stress on the rigorous logical element in mathematics and ignores the intuitional and other non-rigorous tendencies which form an important element in the great bulk of mathematical work, concerning which I shall speak in greater detail later.

IV. *Geometry an Experimental Science.*

Some of you will also regard it as an objection that there are subjects which have almost universally been regarded as branches of mathematics but are excluded by this definition. A striking example of this is geometry, I mean the science of the actual space we live in; for though geometry is, according to Peirce's definition, preëminently a mathematical science, it is not exclusively so. Until a system of axioms is established mathematics cannot begin its work. Moreover the actual perception of spacial relations, not merely in simple cases but in the appreciation of complicated theorems, is an essential element in geometry which has no relation to mathematics as Peirce understands the term. The same is true, to a considerable extent, of such subjects as mechanical drawing and model making, which involve, besides small amounts of physics and mathematics, mainly non-mathematical geometry. Moreover, although the mathematical method is the traditional one for arriving at the truth concerning geometric facts, it is not the only one. Direct appeal to the intuition is often a short and fairly safe cut to geometric results; and on the other hand experiments may be used in geometry, just as they are used every day in physics, to test the truth of a proposition or to determine the value of some geometric magnitude.†

* Cf. the very interesting remarks along this line of C. S. Peirce in *The Monist*, vol. 7, pp. 23–24.

† I am thinking of measurements and observations made on accurately constructed drawings and models. A famous example is Galileo's determination of the area of a cycloid by cutting out a cycloid from a metallic sheet and weighing it.

We must, then, admit, if we hold to Peirce's view, that there is an independent science of geometry just as there is an independent science of physics, and that either of these *may* be treated by mathematical methods. Thus geometry becomes the simplest of the natural sciences, and its axioms are of the nature of physical laws, to be tested by experience and to be regarded as true only within the limits of error of observation. This view, while it has not yet gained universal recognition, should, I believe, prevail, and geometry be recognized as a science independent of mathematics, just as psychology is gradually being recognized as an independent science and not as a branch of philosophy.

The view here set forth, according to which geometry is an experimental science like physics or chemistry, has been held ever since Gauss's time by almost all the leading mathematicians who have been conversant with non-euclidean geometry.* Recently, however, Poincaré has thrown the weight of his great authority against this view,† claiming that the experiments by which it is sought to test the truth of geometric axioms are really not geometric experiments at all but physical ones, and that any failure of these experiments to agree with the ordinary geometric axioms could be explained by the inaccuracy of the *physical* laws ordinarily assumed. There is undoubtedly an important element of truth here. Every experiment depends for its results not merely on the law we wish to test but also on other laws which for the moment we assume to be true. But, if we prefer, we may, in many cases, assume as true the law we were before testing and our experiment will then serve to test some of the remaining laws. If, then, we choose to stick to the ordinary euclidean axioms of geometry in spite of what any future experiments may possibly show, we can do so, but at the cost, perhaps, of our present simple physical laws, not merely in one branch of physics but in several. Poincaré's view‡ is that it will always be expedient to preserve simple geometric laws at all costs, an opinion for which I fail to see sufficient reason.

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* Gauss, Riemann, Helmholtz are the names which will carry perhaps the greatest weight.

† Cf. La science et l'hypothèse. Paris, 1903.

‡ L. c., chapter 5. In particular p. 93.

V. *Kempe's Definition.*

Let us now turn from Peirce's method of defining mathemathics to Kempe's, which, however, I shall present to you in a somewhat modified form.* The point of view adopted here is to try to define mathematics, as other sciences are defined, by describing the objects with which it deals. The diversity of the objects with which mathematics is ordinarily supposed to deal being so great, the first step must be to divest them of what is unessential for the mathematical treatment, and to try in this way to discover their common and characteristic element.

The first point on which Kempe insists is that the objects of mathematical discussion, whether they be the points and lines of geometry, the numbers real or complex of algebra or analysis, the elements of groups or anything else, are always individuals, infinite in number perhaps, but still distinct individuals. In a particular mathematical investigation we may, and usually do, have several different kinds of individuals; as for instance, in elementary plane geometry points, straight lines, and circles. Furthermore we have to deal with certain relations of these objects to one another. For instance, in the example just cited, a given point may or may not lie on a given line; a given line may or may not touch a given circle; three or more points may or may not be collinear, etc. This example shows how in a single mathematical problem a large number of relations may be involved, relations some of which connect two objects, others three, etc. Moreover these relations may connect like or they may connect unlike objects; and finally the order in which the objects are taken is not by any means immaterial in general, as is shown by the relation between three points which states that the third is collinear with and lies between the first two.

But even this is not all; for, besides these objects and relations of various kinds, we often have *operations* by which objects can be combined to yield another object, as, for instance, addition or multiplication of numbers. Here the objects combined and the resulting object are all of the same kind, but this is by no means necessary. We may, for instance, con-

* Kempe has set forth his ideas in rather popular form in the *Proceedings of the London Mathematical Society*, vol. 26 (1894), p. 5; and in *Nature*, vol. 43 (1890), p. 156, where references to his more technical writings will be found.

sider the operation of combining two points and getting the perpendicular bisector of the line connecting them; or we may combine a point and a line and get the perpendicular dropped from the point on the line.

These few examples show how diverse the relations and operations, as well as the objects of mathematics seem at first sight to be. Out of this apparent diversity it is not difficult to obtain a very great uniformity by simply restating the facts in a little different language. We shall find it convenient to indicate that the objects $a, b, c, \cdots$, taken in the order named, satisfy a relation R by simply writing $R(a, b, c, \cdots)$, where it should be understood that among the objects $a, b, c, \cdots$ the same object may occur a number of times. On the other hand if two objects a and b are combined to yield a third object c we may write $a \circ b = c$,* where the symbol $\circ$ is characteristic of the special operation with which we are concerned.

Let us first notice that the equation $a \circ b = c$ denotes merely that the three objects a, b, c bear a certain relation to one another, say $R(a, b, c)$. In other words the idea of an operation or law of combination between the objects we deal with, however convenient and useful it may be as a matter of notation, is essentially merely a way of expressing the fact that the objects combined bear a certain relation to the object resulting from their combination. Accordingly, in a purely abstract discussion like the present, where questions of practical convenience are not involved, we need not consider such rules of combination.†

Furthermore it is easy to see that when we speak of objects of different *kinds*, as for instance the points and lines of geom-

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* I speak here merely of dyadic operations, — i. e, of operations by which two objects are combined to yield a third, — these being by far the most important as well as the simplest. What is said, however, obviously applies to operations by which any number of objects are combined.

† Even from the point of view of the technical mathematician it may sometimes be desirable to adopt the point of view of a relation rather than that of an operation. This is seen, for instance, in laying down a system of postulates for the theory of abstract groups (cf., for example, Huntington, BULLETIN, June, 1902), where the postulate:

If a and b belong to the class, $a \circ b$ belongs to the class,

which in this form looks indecomposable, immediately breaks up, when stated in the relational form, into the following two:

1. If a and b belong to the class, there exists an element c of the class such that $R(a, b, c)$.
2. If a, b, c, d belong to the class, and if $R(a, b, c)$ and $R(a, b, d)$, then $c = d$.

etry, we are introducing a notion which can very readily be expressed in our relational notation. For this purpose we need merely to introduce a further relation which is satisfied by two or more objects when and only when they are of the same "kind."

Let us turn finally to the relations themselves. It is customary to distinguish here between dyadic relations, triadic relations, etc., according as the relation in question connects two objects, three objects, etc. There are, however, relations which may connect any number of objects, as, for instance, the relation of collinearity which may hold between any number of points. Any relation holds for certain ordered groups of objects but not for others, and it is in no way *necessary* for us to fix our attention on the fact, if it be true, that the number of objects in all the groups for which a particular relation holds is the same. This is the point of view we shall adopt, and we shall relegate the property that a relation is dyadic, triadic, etc., to the background along with the various other properties relations may have,* all of which must be taken account of in the proper place.

We are thus concerned in any mathematical investigation from our present point of view with just two conceptions: first a set, or as the logicans say, a *class* of objects $a, b, c, \cdots$; and secondly a class of relations $R, S, T, \cdots$. We may suppose these objects divested of any qualitative, quantitative, special or other attributes which they may have had, and regard them merely as satisfying or not satisfying the relations in question, where, again, we are wholly indifferent to the nature which these relations originally had. And now we are in a position to state what I conceive to be really the essential point in Kempe's definition of mathematics, although I have omitted one of the points on which he insists most strongly, † by saying:

If we have a certain class of objects and a certain class of relations, and if the only questions which we investigate are whether ordered groups of these objects do or do not satisfy the relations, the results of the investigation are called mathematics.

* For instance the property of symmetry. A relation is said to be symmetric if it holds or fails to hold independently of the *order* in which the objects are taken.

† Namely that the only relation that need be considered is that of being "indistinguishable," i. e., a symmetric and transitive relation between two groups of objects.

It is convenient to have a term to designate a class of objects associated with a class of relations between these objects. Such an aggregate we will speak of as a *mathematical system*. If now we have two different mathematical systems, and if a one-to-one correspondence can be set up between the two classes of objects, and also between the two classes of relations in such a way that whenever a certain ordered set of objects of the first system satisfies a relation of that system, the set consisting of the corresponding objects of the second system satisfies the corresponding relation of that system, and *vice versa*, then it is clear that the two systems are, from our present point of view, mathematically equivalent, however different the nature of the objects and relations may be in the two cases.* To use a technical term, the two systems are *simply isomorphic*.†

It will be noticed that in the definition of mathematics just given nothing is said as to the method by which we are to ascertain whether or not a given relation holds between the objects of a given set. The method used may be a purely empirical one, or it may be partly or wholly deductive. Thus, to take a very simple case, suppose our class of objects to consist of a large number of points in a plane, and suppose the only relation between them with which we are concerned is that of collinearity. Then, if the points are given us by being marked in ink on a piece of white paper, we can begin by taking three pins, sticking them into the paper at three of the points; then, by sighting along them, we can determine whether or not these points are collinear. We can do the same with other groups of three points, then with all groups of four points, etc. The same result can be obtained with much less labor if we make use of certain simple properties which the relation of collinearity satisfies, properties which are expressed by such propositions as:

* The point of view here brought out, including the term isomorphism, was first developed in a special case — the theory of groups.

† Inasmuch as the relations in a mathematical system are themselves objects, we may, if we choose, take our class of objects so as to include these relations as well as what we called objects before, some of which, we may remark in passing, may themselves be relations. Looked at from this point of view we need one additional relation which is now the only one which we explicitly call a relation. If we denote this relation by enclosing the objects which satisfy it in parentheses, then if the relation denoted before by $R\ (a, b)$ is satisfied we should now write (R, a, b), whereas we should *not* have (a, R, b) (S, R, a, b), etc. Thus we see that any mathematical system may be regarded as consisting of a class of objects and a *single* relation between them.

$R(a, b, c)$ implies $R(b, a, c)$,
$R(a, b, c, d)$ implies $R(a, b, c)$,
$R(a, b, c)$ and $R(a, b, d)$ together imply $R(a, b, c, d)$, etc.

By means of a small number of propositions of this sort it is easy to show that no empirical observations as to the collinearity of groups of more than three points need be made, and that it may not be necessary to examine even all groups of three points. Having made this relatively small number of observations, the remaining results would be obtained deductively. Finally, we may suppose the points given by their coördinates, in which case the complete answer to our question may be obtained by the purely deductive method of analytic geometry.

According to the modified form of Kempe's definition which I have just stated, mathematics is not necessarily a deductive science. This view, while not in accord with the prevailing ideas of mathematicians, undoubtedly has its advantages as well as its dangers. The non-deductive processes, of which I shall have more to say presently, play too important a part in the life of mathematics to be ignored, and the definition just given has the merit of not excluding them. It would seem, however, that the definition in the form just given is too broad. It would include, for instance, the determination by experimental methods of what pairs of chemical compounds of the known elements react on one another when mixed under given conditions.

VI. *Axioms and Postulates. Existence Theorems.*

If, however, we restrict ourselves to exact or deductive mathematics, it will be seen that Kempe's definition becomes coëxtensive with Peirce's. Here, in order to have a starting-point for deductive reasoning, we must assume a certain number of facts or *primitive propositions* concerning any mathematical system we wish to study, of which all other propositions will be necessary consequences.* We touch here on a subject whose origin goes back to Euclid and which has of late years received great development, primarily at the hands of Italian mathematicians.†

* These primitive propositions may be spoken of as *axioms* or *postulates*, according to the point of view we wish to take concerning their source, the word axiom, which has been much misused of late, indicating an intuitional or empirical source.

† Peano, Pieri, Padoa, Burali-Forti. We may mention here also Hilbert, who, apparently without knowing of the important work of his Italian predecessors, has also done valuable work along these lines.

It is important for us to notice at this point that not merely these primitive propositions but all the propositions of mathematics may be divided into two great classes. On the one hand, we have propositions which state that certain specified objects satisfy certain specified relations. On the other hand are the *existence theorems* which state that there exist objects satisfying, along with certain specified objects, certain specified relations.* These two classes of propositions are well known to logicians and are designated by them *universal* and *particular* propositions respectively.† It is only during the last fifty years or so that mathematicians have become conscious of the fundamental importance in their science of existence theorems which until then they had frequently assumed tacitly as they needed them, without always being conscious of what they were doing.

It is sometimes held by non-mathematicians that if mathematics were really a purely deductive science, it could not have gained anything like the extent which it has without losing itself in trivialities and becoming, as Poincaré puts it, a vast tautology.‡ This view would doubtless be correct if all primitive propositions were universal propositions. One of the most characteristic features of mathematical reasoning, however, is the use which it makes of auxiliary elements. I refer to the auxiliary points and lines in proofs by elementary geometry, the quantities formed by combining in various ways the numbers which enter into the theorems to be proved in algebra, etc. Without the use of such auxiliary elements mathematicians would be incapable of advancing a step; and whenever we make use of such an element in a proof, we are in reality using an existence-theorem.§ These existence theorems need

* Or we might conceivably have existence theorems which state that there exist relations which are sati-fied by certain specified objects; or these two kinds of existence theorems might be combined. If we take the point of view explained in the footnote on p. 128, all existence theorems will be of the type mentioned in the text.

† "All men are mortals" is a standard example of a universal proposition; while as an illustration of a particular proposition is often given: "Some men are Greeks" That this is really an existence theorem is seen more clearly when we state it in the form: "There exists at least one man who is a Greek."

‡ Cf. La Science et l'hypothèse, p. 10.

§ Even when in algebra we consider the sum of two numbers $a+b$, we are using the existence theorem which says that, any two numbers a and b being given, there exists a number c which stands to them in the relation which we indicate in ordinary language by saying that c is the sum of a and b.

not, to be sure, be among the primitive propositions; but if not they must be deduced from primitive propositions some of which are existence-theorems, for it is clear that an existence theorem cannot be deduced from universal propositions alone.* Thus it may fairly be said that existence theorems form the vital principle of mathematics, but these in turn, it must be remembered, would be impotent without the material basis of universal propositions to work upon.

VII. *Russell's Definition.*

We have so far arrived at the view that exact mathematics is the study by deductive methods of what we have called a mathematical system, that is a class of objects and a class of relations between them. If we elaborate this position in two directions we shall reach the standpoint of Russell.†

In the first place Russell makes precise the term *deductive method* by laying down explicitly a list of logical conceptions, and principles which alone are to be used; and secondly he is not willing that mathematics—at least pure mathematics—should be, as Peirce would have it, a science of hypotheses which is not concerned with the truth or falsity of its premises.‡ He insists, on the contrary, that no mathematical system, to use again the technical term introduced above, be studied in pure mathematics whose existence cannot be established from the logical principles on which all mathematics is based. Inasmuch as the development of mathematics during the last fifty years has shown that the existence of most, if not all the mathematical systems which have proved to be important can be deduced when once the existence of positive integers is granted, the point about which interest must centre here is the proof, which Russell attempts, of the existence of this latter system.§

* The power which resides in the method of mathematical induction, so-called, comes from the fact that this method depends on an existence theorem. It is, however, not the only fertile principle in mathematics as Poincaré would have us believe (cf. La science et l'hypothèse). In fact there are great branches of mathematics like elementary geometry, in which it takes little or no part.

† The Principles of Mathematics, Cambridge, England, 1903.

‡ In the formal definition of mathematics at the beginning of the book this is not stated or in any way implied; and yet it comes out so clearly throughout the book that this is a point of view which the author regards as essential, that I have not hesitated to include it as a part of his definition.

§ Cf. also Burali-Forti, Congrès internationale de philosophie. Paris, vol. 3, p. 289.

This proof will necessarily require that, among the logical principles assumed, existence theorems be found. Such theorems do not seem to be explicitly stated by Russell, the existence theorems which make their appearance further on being evolved out of somewhat vague philosophical reasoning. There are also other reasons, into which I cannot enter here, why I am not able to regard the attempt made in this direction by Russell as completely successful.* Nevertheless, in view of the fact that the system of finite positive integers is necessary in almost all branches of mathematics (we cannot speak of a triangle or a hexagon without having the numbers three and six at our disposal), it seems extremely desirable that the system of logical principles which we lay at the foundation of all mathematics be assumed, if possible, broad enough so that the existence of positive integers — at least finite integers — follows from it; and there seems little doubt that this can be done in a satisfactory manner. When this has been done we shall perhaps be able to regard, with Russell, pure mathematics as consisting exclusively of deductions by "logical principles from logical principles."

VIII. *The Non-Deductive Elements in Mathematics.*

I fear that many of you will think that what I have been saying is of an extremely one-sided character, for I have insisted merely on the rigidly deductive form of reasoning used and the purely abstract character of the objects considered in mathematics. These, to the great majority of mathematicians, are only the dry bones of the science. Or, to change the simile, it may perhaps be said that instead of inviting you to a feast I have merely shown you the empty dishes and explained how the feast would be served if only the dishes were filled.† I fully agree with this opinion, and can only plead in excuse that my subject was the *fundamental* conceptions and methods of mathematics, not the infinite variety of detail and application which give our science its real vitality. In fact I should like to subscribe most heartily to the view that in mathematics, as elsewhere, the

* Cf. a paper read by the present writer before the meeting of the American Mathematical Society three days before the present address was delivered, in which this question was discussed. Russell's unequivocal repudiation of nominalism in mathematics seems to me a serious if not an insurmountable barrier to progress.

† Notice that just as the empty dishes could be filled by a great variety of viands, so the empty symbols of mathematics can be given meanings of the most varied sorts.

discussion of such fundamental matters derives its interest mainly from the importance of the theory of which they are the so-called foundations.* I like to look at mathematics almost more as an art than as a science; for the activity of the mathematician, constantly creating as he is, guided though not controlled by the external world of the senses, bears a resemblance, not fanciful I believe but real, to the activity of an artist, of a painter let us say. Rigorous deductive reasoning on the part of the mathematician may be likened here to technical skill in drawing on the part of the painter. Just as no one can become a good painter without a certain amount of this skill, so no one can become a mathematician without the power to reason accurately up to a certain point. Yet these qualities, fundamental though they are, do not make a painter or a mathematician worthy of the name, nor indeed are they the most important factors in the case. Other qualities of a far more subtle sort, chief among which in both cases is imagination, go to the making of the good artist or good mathematician. I must content myself by merely recalling to you this somewhat vague and difficult though interesting field of speculation which arises when we attempt to attach *value* to mathematical work, a field which is familiar enough to us all in the analogous case of artistic or literary criticism.

We are in the habit of speaking of logical rigor and the consideration of axioms and postulates as the foundations on which the superb structure of modern mathematics rests; and it is often a matter of wonder how such a great edifice can rest securely on such a small foundation. Moreover these foundations have not always seemed so secure as they do at present. During the first half of the nineteenth century certain mathematicians of a critical turn of mind — Cauchy, Abel, Weierstrass, to mention the greatest of them — perceived to their dismay that these foundations were not sound, and some of the best efforts of their lives were devoted to strengthening and improving them. And yet I doubt whether the great results of mathematics seemed less certain to any of them because of the weakness they perceived in the foundations on which these

*Cf. the following remark by Study, *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 11 (1902), p. 313:

"So wertvoll auch Untersuchungen über die systematische Stellung der mathematischen Grundbegriffe sind . . . *wertvoller* ist doch noch der materielle Inhalt der einzelnen Disciplinen, um dessentwillen allein ja derartige Untersuchungen überhaupt Zweck haben. . . ."

results are built up. The fact is that what we call mathematical rigor is merely one of the foundation stones of the science; an important and essential one surely, yet not the only thing upon which we can rely. A science which has developed along such broad lines as mathematics, with such numerous relations of its parts both to one another and to other sciences, could not long contain serious error without detection. This explains how, again and again, it has come about, that the most important mathematical developments have taken place by methods which cannot be wholly justified by our present canons of mathematical rigor, the logical "foundation" having been supplied only long after the superstructure had been raised. A discussion and analysis of the non-deductive methods which the creative mathematician really uses would be both interesting and instructive. Here I must content myself with the enumeration of a few of them.

First and foremost there is the use of intuition, whether geometric, mechanical, or physical. The great service which this method has rendered and is still rendering to mathematics both pure and applied is so well known that a mere mention is sufficient.

Then there is the method of experiment; not merely the physical experiments of the laboratory or the geometric experiments I had occasion to speak of a few minutes ago, but also arithmetical experiments, numerous examples of which are found in the theory of numbers and in analysis. The mathematicians of the past frequently used this method in their printed works. That this is now seldom done must not be taken to indicate that the method itself is not used as much as ever.

Closely allied to this method of experiment is the method of analogy which assumes that something true of a considerable number of cases will probably be true in analogous cases. This is, of course, nothing but the ordinary method of induction. But in mathematics induction may be employed not merely in connection with the experimental method, but also to extend results won by deductive methods to other analogous cases. This use of induction has often been unconscious and sometimes overbold, as, for instance, when the operations of ordinary algebra were extended without scruple to infinite series.

Finally there is what may perhaps be called the method of optimism which leads us either wilfully or instinctively to shut

our eyes to the possibility of evil. Thus the optimist who treats a problem in algebra or analytic geometry will say, if he stops to reflect on what he is doing: "I know that I have no right to divide by zero; but there are so many other values which the expression by which I am dividing might have that I will assume that the Evil One has not thrown a zero in my denominator this time." This method, if a proceeding often unconscious can be called a method, has been of great service in the rapid development of many branches of mathematics, though it may well be doubted whether in a subject as highly developed as is ordinary algebra it has not now survived its usefulness.*

While no one of these methods can in any way compare with that of rigorous deductive reasoning as a method upon which to base mathematical results, it would be merely shutting one's eyes to the facts to deny them their place in the life of the mathematical world, not merely of the past but of today. There is now, and there always will be room in the world for good mathematicians of every grade of logical precision. It is almost equally important that the small band whose chief interest lies in accuracy and rigor should not make the mistake of despising the broader though less accurate work of the great mass of their colleagues; as that the latter should not attempt to shake themselves wholly free from the restraint the former would put upon them. The union of these two tendencies in the same individuals, as it was found, for instance, in Gauss and Cauchy, seems the only sure way of avoiding complete estrangement between mathematicians of these two types.

*Cf. the very suggestive remarks by Study, *Jahresbericht d. Deutschen Mathematiker-Vereinigung*, vol. 11 (1902), p. 100, footnote, in which it is pointed out how rigor, in cases of this sort, may not merely serve to increase the correctness of the result, but actually to suggest new fields for mathematical investigation.

THE HISTORY OF MATHEMATICS IN THE NINETEENTH CENTURY.

ADDRESS DELIVERED BEFORE THE DEPARTMENT OF MATHEMATICS OF THE INTERNATIONAL CONGRESS OF ARTS AND SCIENCE, ST. LOUIS, SEPTEMBER 20, 1904.

BY PROFESSOR JAMES PIERPONT.

The extraordinary development of mathematics in the last century is quite unparalleled in the long history of this most ancient of sciences. Not only have those branches of mathematics which were taken over from the eighteenth century steadily grown but entirely new ones have sprung up in almost bewildering profusion, and many of these have promptly assumed proportions of vast extent.

As it is obviously impossible to trace in the short time allotted to me the history of mathematics in the nineteenth century, even in merest outline, I shall restrict myself to the consideration of some of its leading theories.

Theory of Functions of a Complex Variable.

Without doubt one of the most characteristic features of mathematics in the last century is the systematic and universal use of the complex variable. Most of the great mathematical theories received invaluable aid from it, and many owe to it their very existence. What would the theory of differential equations or elliptic functions be to-day without it, and is it probable that Poncelet, Steiner, Chasles, and von Staudt would have developed synthetic geometry with such elegance and perfection without its powerful stimulus?

The necessities of elementary algebra kept complex numbers persistently before the eyes of every mathematician. In the eighteenth century the more daring, as Euler and Lagrange, used them sparingly; in general one avoided them when possible. Three events, however, early in the nineteenth century changed the attitude of mathematicians toward this mysterious guest. In 1813–14 Argand published his geometric interpretation of complex numbers. In 1824 came the discovery by Abel of the imaginary period of the elliptic function. Finally Gauss in his

second memoir on biquadratic residues (1832) proclaims them a legitimate and necessary element of analysis.

The theory of functions of a complex variable may be said to have had its birth when Cauchy discovered his integral theorem

$$\int_c f(x)dx = 0,$$

published in 1825. In a long series of publications beginning with the Cours d'analyse, 1821, Cauchy gradually developed his theory of functions and applied it to problems of the most diverse nature; e. g., existence theorems for implicit functions and the solutions of certain differential equations, the development of functions in infinite series and products, and the periods of integrals of one-valued and many-valued functions.

Meanwhile Germany is not idle; Weierstrass and Riemann develop Cauchy's theory along two distinct and original paths. Weierstrass starts with an explicit analytic expression, a power series, and defines his function as the totality of its analytical continuations. No appeal is made to geometric intuition, his entire theory is strictly arithmetical. Riemann, growing up under Gauss and Dirichlet, not only relies largely on geometric intuition, but also does not hesitate to impress mathematical physics into his service. Two noteworthy features of his theory are the many-leaved surfaces named after him, and the extensive use of conformal representation.

The history of functions as first developed is largely a theory of algebraic functions and their integrals. A general theory of functions is only slowly evolved. For a long time the methods of Cauchy, Riemann, and Weierstrass were cultivated along distinct lines by their respective pupils. The schools of Cauchy and Riemann were the first to coalesce. The entire rigor which has recently been imparted to their methods has removed all reason for founding, as Weierstrass and his school have urged, the theory of functions on a single algorithm, viz., the power series. We may therefore say that at the close of the century there is only one theory of functions, in which the ideas of its three great creators are harmoniously united.

Let us note briefly some of its lines of advance. Weierstrass early observed that an analytic expression might represent different analytic functions in different regions. Associated with this is the phenomenon of natural boundaries. The question

therefore arose as to what is the most general domain of definition of an analytic function. Runge has shown that any connected region may serve this purpose. An important line of investigation relates to the analytic expression of a function by means of infinite series, products, and fractions. Here may be mentioned Weierstrass's discovery of prime factors; the theorems of Mittag-Leffler and Hilbert; Poincaré's uniformization of algebraic and analytic functions by means of a third variable, and the work of Stieltjes, Pincherle, Padé, and Van Vleck on infinite fractions.

Since an analytic function is determined by a single power series, which in general has a finite circle of convergence, two problems present themselves: to determine 1) the singular points of the analytic function so defined, and 2) an analytic expression valid for its whole domain of definition. The celebrated memoir of Hadamard inaugurated a long series of investigations on the first problem; while Mittag-Leffler's star theorem is the most important result yet obtained relating to the second.

Another line of investigation relates to the work of Poincaré, Borel, Stieltjes, and others on divergent series. It is indeed a strange vicissitude of our science that these series, which early in the century were supposed to be banished once and for all from rigorous mathematics, should at its close be knocking at the door for readmission.

Let us finally note an important series of memoirs on integral transcendental functions beginning with Weierstrass, Laguerre, and Poincaré.

Algebraic Functions and Their Integrals.

A branch of the theory of functions has been developed to such an extent that it may be regarded as an independent theory, we mean the theory of algebraic functions and their integrals. The brilliant discoveries of Abel and Jacobi in the elliptic functions from 1824 to 1829 prepared the way for a similar treatment of the hyperelliptic case. Here a difficulty of gravest nature was met. The corresponding integrals have $2p$ linearly independent periods; but, as Jacobi had shown, a one-valued function having more than two periods admits a period as small as we choose. It therefore looked as if the elliptic functions admitted no further generalization. Guided

by Abel's theorem, Jacobi at last discovered the solution to the difficulty, 1832; to get functions analogous to the elliptic functions we must consider functions not of one but of p independent variables, viz., the p independent integrals of the first species. The great problem now before mathematicians, known as Jacobi's problem of inversion, was to extend this aperçu to the case of any algebraic configuration and develop the consequences. The first to take up this immense task were Weierstrass and Riemann, whose results belong to the most brilliant achievements of the century. Among the important notions hereby introduced we note the following: the birational transformation, rank of an algebraic configuration, class invariants, prime functions, the theta and multiply periodic functions in several variables. Of great importance is Riemann's method of proving existence theorems as also his representation of algebraic functions by means of integrals of the second species.

A new direction was given to research in this field by Clebsch, who considered the fundamental algebraic configuration as defining a curve. His aim was to bring about a union of Riemann's ideas and the theory of algebraic curves for their mutual benefit. Clebsch's labors were continued by Brill and Noether; in their work the transcendental methods of Riemann are placed quite in the background. More recently Klein and his school have sought to unite the transcendental methods of Riemann with the geometric direction begun by Clebsch, making systematic use of homogeneous coördinates and the invariant theory. Noteworthy also is Klein's use of normal curves in $(p-1)$-way space to represent the given algebraic configuration. Dedekind and Weber, Hensel and Landsberg have made use of the ideal theory with marked success. Many of the difficulties of the older theory, for example the resolution of singularities of the algebraic configuration, are treated with a truly remarkable ease and generality.

In the theory of multiply periodic functions and the general θ functions we mention, besides those of Weierstrass, the researches of Prym, Krazer, Frobenius, Poincaré, and Wirtinger.

Automorphic Functions.

Closely connected with the elliptic functions is a class of functions which has come into great prominence in the last quarter of a century, viz.: the elliptic modular and automorphic

functions. Let us consider first the modular functions of which the modulus κ and the absolute invariant J are the simplest types.

The transformation theory of Jacobi gave algebraic relations between such functions in endless number. Hermite, Fuchs, Dedekind and Schwarz are forerunners, but the theory of modular functions as it stands to-day is principally due to Klein and his school. Its goal is briefly stated thus: To determine all subgroups of the linear group

$$x' = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad (1)$$

where $\alpha, \beta, \gamma, \delta$ are integers and $\alpha\delta - \beta\gamma \neq 0$; to determine for each such group associate modular functions and to investigate their relation to one another and especially to J. Important features in this theory are the congruence groups of (1); the fundamental polygon belonging to a given subgroup, and its use as substitute for a Riemann surface; the principle of reflection on a circle; the modular forms.

The theory of automorphic functions is due to Klein and Poincaré. It is a generalization of the modular functions; the coefficients in (1) being any real or imaginary numbers, with non-vanishing determinant, such that the group is discontinuous. Both authors have recourse to non-euclidean geometry to interpret the substitutions (1). Their manner of showing the existence of functions belonging to a given group is quite different. Poincaré by a brilliant stroke of genius actually writes down their arithmetic expressions in terms of his celebrated θ series. Klein employs the existence methods of Riemann. The relation of automorphic functions to differential equations is studied by Poincaré in detail. In particular, he shows that both variables of a linear differential equation with algebraic coëfficients can be expressed uniformly by their means.

Differential Equations.

Let us turn now to another great field of mathematical activity, the theory of differential equations. The introduction of the theory of functions has completely revolutionized this subject. At the beginning of the nineteenth century many important results had indeed been established, particularly by Euler and Lagrange; but the methods employed were artificial,

and broad comprehensive principles were lacking. By various devices one tried to express the solution in terms of the elementary functions and quadratures — a vain attempt; for, as we know now, the goal they strove so laboriously to reach was in general unattainable.

A new epoch began with Cauchy, who by means of his new theory of functions first rigorously established the existence of the solution of certain classes of equations in the vicinity of regular points. He also showed that many of the properties of the elliptic functions might be deduced directly from their differential equations. Ere long, the problem of integrating a differential equation changed its base. Instead of seeking to express its solution in terms of the elementary functions and quadratures, one asked what is the nature of the functions defined by a given equation. To answer this question we must first know what are the singular points of the integral function and how it behaves in their vicinity. The number of memoirs on this fundamental and often difficult question is enormous; but this is not strange if we consider the great variety of interesting and important classes of equations which have to be studied.

One of the first to open up this new path was Fuchs, whose classic memoirs (1866–68) gave the theory of linear differential equations its birth. These equations enjoy a property which renders them particularly accessible, viz., the absence of movable singular points. They may, however, possess points of indetermination, to use Fuchs's terminology, and little progress has been made in this case. Noteworthy in this connection is the introduction by von Koch of infinite determinants, whose importance was first shown by our distinguished countryman, Hill; also the use of divergent series — that invention of the devil, as Abel called them — by Poincaré. A particular class of linear differential equations of great importance is the hypergeometric equation; the results obtained by Gauss, Kummer, Riemann, and Schwarz relating to this equation have had the greatest influence on the development of the general theory. The great extent of the theory of linear differential equations may be estimated when we recall that within its borders it embraces not only almost all the elementary functions, but also the modular and automorphic functions.

Too important to pass over in silence is the subject of algebraic differential equations with uniform solutions. The brilliant researches of Painlevé deserve especial mention.

Another field of great importance, especially in mathematical physics, relates to the determination of the solution of differential equations with assigned boundary conditions. The literature of this subject is enormous; we may therefore be pardoned if mention is made only of the investigations of Green, Sturm, Liouville, Bôcher, Riemann, Schwarz, C. Neumann, Poincaré, and Picard.

Since 1870 the theory of differential equations has been greatly advanced by Lie's theory of groups. Assuming that an equation or a system of equations admits one or more infinitesimal transformations, Lie has shown how these may be employed to simplify the problem of integration. In many cases they give us exact information how to conduct the solution and upon what system of auxiliary equations the solution depends. One of the most striking illustrations of this is the theory of ordinary linear differential equations which Picard and Vessiot have developed, analogous to Galois's theory for algebraic equations. An interesting result of this theory is a criterion for the solution of such equations by quadratures. As an application we find that Riccati's equation cannot be solved by quadratures. The attempts to effect such a solution of this celebrated equation in the preceding century were therefore necessarily in vain.

A characteristic feature of Lie's theories is the prominence given to the geometrical aspects of the questions involved. Lie thinks in geometrical images, the analytical formulation comes afterwards. Already Monge had shown how much might be gained by geometrizing the problem of integration. Lie has gone much farther in this direction. Besides employing all the geometrical notions of his predecessors extended to n-way space, he has introduced a variety of new conceptions, chief of which are his surface element and contact transformations.

He has also used with great effect Plücker's line geometry and his own sphere geometry in the study of certain types of partial differential equations of the first and second orders which are of great geometrical interest, for example equations whose characteristic curves are lines of curvature, geodesics, etc. Let us close by remarking that Lie's theories not only afford new and valuable points of view for attacking old problems but also give rise to a host of new ones of great interest and importance.

Groups.

We turn now to the second dominant idea of the century, the group concept.

Groups first became objects of study in algebra when Lagrange 1770, Ruffini 1799, and Abel 1826 employed substitution groups with great advantage to their work on the quintic. The enormous importance of groups in algebra was, however, first made clear by Galois, whose theory of the solution of algebraic equations is one of the great achievements of the century. Its influence has stretched far beyond the narrow bounds of algebra.

With an arbitrary but fixed domain of rationality, Galois observed that every algebraic equation has attached to it a certain group of substitutions. The nature of the auxiliary equations required to solve the given equation is completely revealed by an inspection of this group.

Galois's theory showed the importance of determining the sub-groups of a given substitution group, and this problem was studied by Cauchy, Serret, Mathieu, Kirkman and others. The publication of Jordan's great treatise in 1870 is a noteworthy event. It collects and unifies the results of his predecessors and contains an immense amount of new matter.

A new direction was given to the theory of groups by the introduction by Cayley of abstract groups (1854, 1878). The work of Sylow, Hölder, Frobenius, Burnside, Cole, and Miller deserves especial notice.

Another line of researches relates to the determination of the finite groups in the linear group of any number of variables. These groups are important in the theory of linear differential equations with algebraic solutions; in the study of certain geometrical problems, as the points of inflection of a cubic, the 27 lines on a surface of the third order; in crystallography, etc. They also enter prominently in Klein's Formenproblem. An especially important class of finite linear groups are the congruence groups first considered by Galois. Among the laborers in the field of linear groups we note Jordan, Klein, Moore, Maschke, Dickson, Frobenius, and Wiman.

Up to the present we have considered only groups of finite order. About 1870 entirely new ideas, coming from geometry and differential equations, give the theory of groups an unexpected development. Foremost in this field are Lie and Klein.

Lie discovers and gradually perfects his theory of continuous transformation groups and shows their relations to many different branches of mathematics. In 1872 Klein publishes his Erlanger Programm and in 1877 begins his investigations on

elliptic modular functions, in which infinite *discontinuous* groups are of primary importance, as we have already seen. In the now famous Erlanger Programm, Klein asks what is the principle which underlies and unifies the heterogeneous geometric methods then in vogue, as for example, the geometry of the ancients whose figures are rigid and invariable, the modern projective geometry, whose figures are in ceaseless flux, passing from one form to another, the geometries of Plücker and Lie in which the elements of space are no longer points but lines, spheres, or other configurations at pleasure, the geometry of birational transformations, the analysis situs, etc., etc. Klein finds this answer: In each geometry we have a system of objects and a group which transforms these objects one into another. We seek the invariants of this group. In each case it is the abstract group which is essential, and not the concrete objects. The fundamental role of a group in geometrical research is thus made obvious. Its importance in the solution of algebraic equation, in the theory of differential equations, in the automorphic functions we have already seen. The immense theory of algebraic invariants developed by Cayley and Sylvester, Aronhold, Clebsch, Gordan, Hermite, Brioschi, and a host of zealous workers in the middle of the century, also finds its place in the far more general invariant theory of Lie's theory of groups. The same is true of the theory of surfaces as far as it rests on the theory of differential forms. In the theory of numbers, groups have many important applications, for example, in the composition of quadratic forms and the cyclotomic bodies. Finally let us note the relation between hypercomplex numbers and continuous groups discovered by Poincaré.

In resumé, we may thus say that the group concept, hardly noticeable at the beginning of the century, has at its close become one of the fundamental and most fruitful notions in the whole range of our science.

Infinite Aggregates.

Leaving the subject of groups, we consider now briefly another fundamental concept, viz., infinite aggregates. In the most diverse mathematical investigations we are confronted with such aggregates. In geometry the conceptions of a curve, surface, regon, frontier, etc., when examined carefully, lead us to a rich

variety of aggregates. In analysis they also appear, for example the domain of definition of an analytical function, the points where a function of a real variable ceases to be continuous or to have a differential coefficient, the points where a series of functions ceases to be uniformly convergent, etc.

To say that an aggregate (not necessarily a point aggregate) is infinite is often an important step; but often again only the first step. To penetrate farther into the problem may require us to state *how* infinite. This requires us to make distinctions in infinite aggregates, to discover fruitful principles of classification, and to investigate the properties of such classes.

The honor of having done this belongs to Georg Cantor. The theory of aggregates is for the most part his creation; it has enriched mathematical science with fundamental and far reaching notions and results.

The theory falls into two parts; a theory of aggregates in general, and a theory of point aggregates. In the theory of point aggregates the notion of limiting points gives rise to important classes of aggregates, as discrete, dense, complete, perfect, connected, etc., which are so important in the function theory.

In the general theory two notions are especially important, viz.: the one-to-one correspondence of the elements of two aggregates, and well ordered aggregates. The first leads to cardinal numbers and the idea of enumerable aggregates, the second to transfinite or ordinal numbers.

Two striking results of Cantor's theory may be mentioned: the algebraic and therefore the rational numbers, although everywhere dense, are enumerable; and secondly, one-way and n-way space have the same cardinal number.

Cantor's theory has already found many applications, especially in the function theory, where it is today an indispensable instrument of research.

Functions of Real Variables. The Critical Movement.

One of the most conspicuous and distinctive features of mathematical thought in the nineteenth century is its critical spirit. Beginning with the calculus, it soon permeates all analysis, and toward the close of the century it overhauls and recasts the foundations of geometry and aspires to further conquests in mechanics and in the immense domains of mathematical physics.

Ushered in with Lagrange and Gauss just at the close of the eighteenth century, the critical movement receives its first decisive impulse from the teachings of Cauchy, who in particular introduces our modern definition of limit and makes it the foundation of the calculus. We must also mention in this connection Abel, Bolzano, and Dirichlet. Especially Abel adopted the reform ideas of Cauchy with enthusiasm and made important contributions in infinite series.

The figure, however, which towers above all others in this movement, whose name has become a synonym of rigor, is Weierstrass. Beginning at the very foundations, he creates an arithmetic of real and complex numbers, assuming the theory of positive integers to be given. The necessity of this is manifest when we recall that until then the simplest properties of radicals and logarithms were utterly devoid of a rigorous foundation; so for example $\sqrt{2}\,\sqrt{5} = \sqrt{10}$, $\log 2 + \log 5 = \log 10$.

Characteristic of the pre-Weierstrassian era is the loose way in which geometrical and other intuitional ideas were employed in the demonstration of analytic theorems. Even Gauss is open to this criticism. The mathematical world received a great shock when Weierstrass showed them an example of a continuous function without a derivative, and Hankel and Cantor by means of their principle of condensation of singularities could construct analytical expressions for functions having in any interval, however small, an infinity of points of oscillation, an infinity of points in which the differential coefficient is altogether indeterminate, or an infinity of points of discontinuity. Another rude surprise was Cantor's discovery of the one-to-one correspondence between the points of a unit segment and a unit square, followed up by Peano's example of a space filling curve.

These examples and many others made it very clear that the idea of a curve, a surface region, motion, etc., instead of being clear and simple, were extremely vague and complex. Until these notions had been cleared up, their admission in the demonstration of an analytical theorem was therefore not to be tolerated. On a purely arithmetical basis, with no appeal to our intuition, Weierstrass develops his stately theory of functions, which culminates in the theory of abelian and multiply periodic functions.

But the notion of rigor is relative and depends on what we are

willing to assume either tacitly or explicitly. As we observed, Gauss, whose rigor was the admiration of his contemporaries, freely admitted geometrical notions. This Weierstrass would criticise. On the other hand Weierstrass has committed a grave oversight: he nowhere shows that his definitions relative to the numbers he introduces do not involve mutual contradictions. If he replied that such contradictions would involve contradictions in the theory of positive integers, one might ask what assurance have we that such contradictions may not actually exist? A flourishing young school of mathematical logic has recently grown up in Italy under the influence of Peano. They have investigated with marked success the foundations of analysis and geometry, and have in particular endeavored to show the non-contradictoriness of the axioms of our number system by making them depend on the axioms of logic, which axioms we must admit in order to reason at all.

The critical spirit which in the first half of the century was to be found in the writings of only a few of the foremost mathematicians has in the last quarter of the century become almost universal, at least in analysis. A searching examination of the foundations of arithmetic and the calculus has brought to light the insufficiency of much of the reasoning formerly considered as conclusive. It became necessary to build up these subjects anew. The theory of irrational numbers invented by Weierstrass has been supplanted by the more flexible theories of Dedekind and Cantor. Stolz has given us a systematic and rigorous treatment of arithmetic. The calculus has been completely overhauled and arithmetized by Thomae, Harnack, Peano, Stolz, Jordan, and Vallée-Poussin.

Leaving the calculus, let us notice briefly the theory of functions of real variables. The line of demarcation between these two subjects is extremely arbitrary. We might properly place in the latter all those finer and deeper questions relating to the number system, the study of the curve, surface, and other geometrical notions, the peculiarities that functions present with reference to discontinuity, oscillation, differentiation, and integration, as well as a very extensive class of investigations whose object is the greatest possible extension of the processes, concepts, and results of the calculus. Among the many not yet mentioned who have made important contributions to this subject we note: Fourier, Riemann, Stokes, Dini, Tannery, Pringsheim, Arzelà, Osgood, Brodén, Ascoli, Borel, Baire, Köpcke, Hölder, Volterra, and Lebesgue.

Closely related with the differential calculus is the calculus of variations; in the former the variables are given infinitesimal variations, in the latter the functions. Developed in a purely formal manner by Jacobi, Hamilton, Clebsch and others in the first part of the century, a new epoch began with Weierstrass who, having subjected the labors of his predecessors to an annihilating criticism, placed the theory on a new and secure foundation, and so opened the path for further research by Schwarz, A. Mayer, Scheeffer, von Escherich, Kneser, Osgood, Bolza, Kobb, Zermelo and others. At the very close of the century Hilbert has given the theory a fresh impulse by the introduction of new and powerful methods which enable us in certain cases to neglect the second variation and to simplify the consideration of the first.

Theory of Numbers. Algebraic Bodies.

The theory of numbers as left by Fermat, Euler and Legendre was for the most part concerned with the solution of diophantine equations, *i. e.*, given an equation $f(x, y, z, \cdots) = 0$, whose coefficients are integers, to find all rational, and especially all integral solutions. In this problem Lagrange had shown the importance of considering the theory of forms. A new era begins with the appearance of Gauss's Disquisitiones Arithmeticæ in 1801. This great work is remarkable for three things; 1) The notion of divisibility in the form of congruences is shown to be an instrument of wonderful power; 2) the diophantine problem is thrown in the background and the theory of forms is given a dominant role; 3) the introduction of algebraic numbers, viz., the roots of unity.

The theory of forms has been further developed along the lines of the Disquisitiones by Dirichlet, Eisenstein, Hermite, H. J. S. Smith, and Minkowski.

Another part of the theory of numbers also goes back to Gauss, viz., algebraic numerical bodies. The law of reciprocity of quadratic residues, one of the gems of the higher arithmetic, was first rigorously proved by Gauss. His attempts to extend this theorem to cubic and biquadratic residues showed that the elegant simplicity which prevailed in quadratic residues was altogether missing in these higher residues until one passed from the domain of real integers to the domain formed of the third and fourth roots of unity. In these domains, as

Gauss remarked, algebraic integers have essentially the same properties as ordinary integers. Further exploration in this new and promising field by Jacobi, Eisenstein, and others soon brought to light the fact that already in the domain formed of the 23d roots of unity the laws of divisibility were altogether different from those of ordinary integers; in particular a number could be expressed as the product of prime factors in more than one way. Further progress in this direction was therefore apparently impossible.

It is Kummer's immortal achievement to have made further progress possible by the invention of his ideals. These he applied to Fermat's celebrated last theorem and the law of reciprocity of higher residues.

The next step in this direction was taken by Dedekind and Kronecker, who developed the ideal theory for any algebraic domain. So arose the theory of algebraic numerical bodies which has come into such prominence in the last decades of the century through the researches of Hensel, Hurwitz, Minkowski, Weber, and above all Hilbert.

Kronecker has gone farther; in his classic Grundzüge he has shown that similar ideas and methods enable us to develop a theory of algebraic bodies in any number of variables. The notion of divisibility, so important in the preceding theories, is generalized by Kronecker still farther in the shape of his system of moduli.

Another noteworthy field of research opened up by Kronecker is the relation between binary quadratic forms with negative determinant and complex multiplication of elliptic functions. H. J. S. Smith, Gierster, Hurwitz, and especially Weber have made important contributions.

A method of great power in certain investigations has been created by Minkowski which he calls the Geometrie der Zahlen. Introducing a generalization of the distance function, he is led to the conception of a fundamental body (Aichkörper). Minkowski shows that every fundamental body is nowhere concave and conversely to each such body belongs a distance function. A theorem of great importance is now the following: The minimum value which each distance function has at the lattice points is not greater than a certain number depending on the function chosen.

We wish finally to mention a line of investigation which makes use of the infinitesimal calculus and even the theory of

functions. Here belong the brilliant researches of Dirichlet relating to the number of classes of binary forms for a given determinant, the number of primes in a given arithmetic progression; and Riemann's remarkable memoir on the number of primes in a given interval.

On this analytical side of the theory of numbers we notice also the researches of Mertens, Weber, von Mangoldt, and Hadamard.

Projective Geometry.

The tendencies of the eighteenth century were predominantly analytic. Mathematicians were absorbed for the most part in developing the wonderful instrument of the calculus with its countless applications. Geometry made relatively little progress. A new era begins with Monge. His numerous and valuable contributions to analytic, descriptive, and differential geometry and especially his brilliant and inspiring lectures at the Ecole polytechnique (1795–1809) put fresh life in geometry and prepared it for a new and glorious development in the nineteenth century.

When one passes in review the great achievements which have made the nineteenth century memorable in the annals of our science, certainly projective geometry will occupy a foremost place. Pascal, De la Hire, Monge, and Carnot are forerunners, but Poncelet, a pupil of Monge, is its real creator. The appearance of his Traité des propriétés projectives des figures in 1822 gives modern geometry its birth. In it we find the line at infinity, the introduction of imaginaries, the circular points at infinity, polar reciprocation, a discussion of homology, the systematic use of projection, section, and anharmonic ratio.

While the countrymen of Poncelet, especially Chasles, do not fail to make numerous and valuable contributions to the new geometry, the next great steps in advance are made on German soil. In 1827 Möbius publishes the Barycentrische Calcul; Plücker's Analytisch-geometrische Entwickelungen appears in 1828–31; and Steiner's Systematische Entwickelung der Abhängigkeit geometrischer Gestalten von einander in 1832. In the ten years which embrace the publication of these immortal works of Poncelet, Plücker, and Steiner, geometry has made more real progress than in the 2,000 years which had elapsed since the time of Apollonius. The ideas which had

been slowly taking shape since the time of Descartes suddenly crystallized and almost overwhelmed geometry with an abundance of new ideas and principles.

To Möbius we owe the introduction of homogeneous coördinates, and the far reaching conception of geometric transformation including collineation and duality as special cases. To Plücker we owe the use of the abbreviated notation which permits us to study the properties of geometric figures without intervention of the coördinates, the introduction of line and plane coördinates and the notion of generalized space elements. Steiner, who has been called the greatest geometer since Apollonius, besides enriching geometry in countless ways, was the first to employ systematically the method of generating geometric figures by means of projective pencils.

Other noteworthy works belonging to this period are Plücker's System der analytischen Geometrie (1835) and Chasles's classic Aperçu (1837).

Already at this stage we notice a bifurcation in geometric methods. Steiner and Chasles become eloquent champions of the synthetic school of geometry, while Plücker and later Hesse and Cayley are leaders in the analytic movement. The astonishing fruitfulness and beauty of synthetic methods threatened for a short time to drive the analytic school out of existence. The tendency of the synthetic school was to banish more and more metrical methods. In effecting this the anharmonic ratio became constantly more prominent. To define this fundamental ratio without reference to measurement and so to free projective geometry from the galling bondage of metric relations was thus a problem of fundamental importance. The glory of this achievement, which has, as we shall see, a far wider significance, belongs to von Staudt. Another equally important contribution of von Staudt to synthetic geometry is his theory of imaginaries. Poncelet, Steiner, Chasles operate with imaginary elements as if they were real. Their only justification is recourse to the so-called principle of continuity or to some other equally vague principle. Von Staudt gives this theory a rigorous foundation, defining the imaginary points, lines and planes by means of involutions without ordinal elements.

The next great advance made is the advent of the theory of algebraic invariants. Since projective geometry is the study of those properties of geometric figures which remain unaltered

by projective transformations, and since the theory of invariants is the study of those forms which remain unaltered (except possibly for a numerical factor) by the group of linear substitutions, these two subjects are inseparably related and in many respects only different aspects of the same thing. It is no wonder then that geometers speedily applied the new theory of invariants to geometrical problems. Among the pioneers in this direction were Cayley, Salmon, Aronhold, Hesse, and especially Clebsch.

Finally we must mention the introduction of the line as a space element. Forerunners are Grassmann, 1844, and Cayley, 1859, but Plücker in his memoirs of 1865 and his work Neue Geometrie des Raumes (1868–69), was the first to show its great value by studying complexes of the first and second order and calling attention to their application to mechanics and optics.

The most important advance over Plücker has been made by Klein who takes as coördinates six line complexes in involution. Klein also observed that line geometry may be regarded as a point geometry on a quadric in five-way space. Other laborers in this field are Clebsch, Reye, Study, Segre, Sturm, and Koenigs.

Differential Geometry.

During the first quarter of the century this important branch of geometry was cultivated chiefly by the French. Monge and his school study with great success the generation of surfaces in various ways, the properties of envelopes, evolutes, lines of curvature, asymptotic lines, skew curves, orthogonal systems, and especially the relation between the surface theory and partial differential equations.

The appearance of Gauss's Disquisitiones generales circa superficies curvas in 1828 marks a new epoch. Its wealth of new ideas has furnished material for countless memoirs and given geometry a new direction. We find here the parametric representation of a surface, the introduction of curvilinear coördinates, the notion of spherical image, the gaussian measure of curvature, and a study of geodesics. But by far the most important contributions that Gauss makes in this work are the consideration of a surface as a flexible inextensible film or membrane and the importance given quadratic differential forms.

We consider now some of the lines along which differential geometry has advanced. The most important is perhaps the

theory of differential quadratic forms with their associate invariants and parameters. We mention here Lamé, Beltrami, Mainardi, Codazzi, Christoffel, Weingarten, and Maschke.

An especially beautiful application of this theory is the immense subject of applicability and deformation of surfaces in which Minding, Bauer, Beltrami, Weingarten, and Voss have made important contributions.

Intimately related with the theory of applicability of two surfaces is the theory of surfaces of constant curvature which play so important a part in non-euclidean geometry. We mention here the work of Minding, Bonnet, Beltrami, Dini, Bäcklund, and Lie.

The theory of rectilinear congruences has also been the subject of important researches from the standpoint of differential geometry. First studied by Monge as a system of normals to a surface and then in connection with optics by Malus, Dupin, and Hamilton, the general theory has since been developed by Kummer, Ribaucour, Guichard, Darboux, Voss, and Weingarten. An important application of this theory is the infinitesimal deformation of a surface.

Minimum surfaces have been studied by Monge, Bonnet and Enneper. The subject owes its present extensive development principally to Weierstrass, Riemann, Schwarz, and Lie. In it we find harmoniously united the theory of surfaces, the theory of functions, the calculus of variations, the theory of groups, and mathematical physics.

Another extensive division of differential geometry is the theory of orthogonal systems, of such importance in physics. We note especially the investigations of Dupin, Jacobi, Lamé, Darboux, Combescure, and Bianchi.

We have already mentioned the intimate relation between differential geometry and differential equations developed by Monge and Lie. Among the workers in this fruitful field Darboux deserves especial mention.

One of the most original and interesting contributions to geometry in the last decades of the century is Lie's sphere geometry. As a brilliant application of it to differential geometry we may mention the relation discovered by Lie between asymptotic lines and lines of curvature of a surface. The subject of sphere geometry has been developed also by Darboux, Reye, Laguerre, Loria, P. F. Smith, and E. Müller.

Other Branches of Geometry.

Under this head we group a number of subjects too important to pass over in silence, yet which cannot be considered at length for lack of time.

In the first place is the immense subject of algebraic curves and surfaces. Adequately to develop all the important and elegant properties of curves and surfaces of the second order alone, would require a bulky volume. In this line of ideas would follow curves and surfaces of higher order and class. Their theory is far less complete, but this lack it amply makes good by offering an almost bewildering variety of configurations to classify and explore. No single geometer has contributed more to this subject than Cayley.

A theory of great importance is the geometry on a curve or surface inaugurated by Clebsch in 1863. Expressing the coördinates of a plane cubic by means of elliptic functions and employing their addition theorems, he deduced with hardly any calculation Steiner's theorem relating to the inscribed polygons and various theorems concerning conics touching the curve. Encouraged by such successes Clebsch proposed to make use of Riemann's theory of abelian functions in the study of algebraic curves of any order. The most important result was a new classification of such curves. Instead of the linear transformation Clebsch, in harmony with Riemann's ideas, employs the birational transformation as a principle of classification. From this standpoint we ask what are the properties of algebraic curves which remain invariant for such transformation.

Brill and Noether follow Clebsch. Their method is, however, algebraical and rests on their celebrated residual theorem, which in their hands takes the place of Abel's theorem. We mention further the investigations of Castelnuovo, Weber, Krauss and Segre. An important division of this subject is the theory of correspondences. First studied by Chasles for curves of deficiency 0 in 1864, Cayley and immediately after Brill extended the theory to the case of any p. The most important advance made in later years has been made by Hurwitz, who considers the totality of possible correspondences on an algebraic curve, making use of the abelian integrals of the first species.

Alongside the geometry on a curve is the vastly more difficult and complicated geometry on a surface, or more generally, on any algebraic spread in n-way space. Starting from a

remark of Clebsch 1868, Noether made the first great step in his famous memoirs of 1868–74. Further progress has been due to the French and Italian mathematicians. Picard, Poincaré, and Humbert make use of transcendental methods in which figure prominently double integrals which remain finite on the surface and single integrals of total differentials. On the other hand Enriques and Castelnuovo have attacked the subject from a more algebraic-geometric standpoint by means of linear systems of algebraic curves on the surface.

The first invariants of a surface were discovered by Clebsch and Noether; still others have been found by Castelnuovo and Enriques in connection with irregular surfaces.

Leaving this subject let us consider briefly the geometry of n dimensions. A characteristic of nineteenth century mathematics is the generality of its methods and results. When such has been impossible with the elements in hand, fresh ones have been invented; witness the introduction of imaginary numbers in algebra and the function theory, the ideals of Kummer in the theory of numbers, the line and plane at infinity in projective geometry. The benefit that analysis derived from geometry was too great not to tempt mathematicians to free the latter from the narrow limits of three dimensions and so give it the generality that the former has long enjoyed. The first pioneer in this abstract field was Grassmann (1844); we must, however, consider Cayley as the real founder of n-dimensional geometry (1869). Notable contributions have been made by the Italian school, Veronese, Segre, and others.

Non-Euclidean Geometry.

Each century takes over as a heritage from its predecessors a number of problems whose solution previous generations of mathematicians have arduously but vainly sought. It is a signal achievement of the nineteenth century to have triumphed over some of the most celebrated of these problems.

The most ancient of them is the quadrature of the circle, which already appears in our oldest mathematical document, the Papyrus Rhind, B. C. 2000. Its impossibility was finally shown by Lindemann, 1882.

Another famous problem relates to the solution of the quintic, which had engaged the attention of mathematicians since the middle of the sixteenth century. The impossibility of ex-

pressing its roots by radicals was finally shown by the youthful Abel (1824), while Hermite and Kronecker (1858) showed how they might be expressed by the elliptic modular functions, and Klein (1875) by means of the icosahedral irrationality.

But of all problems which have come down from the past by far the most celebrated and important relates to Euclid's parallel axiom. Its solution has profoundly affected our views of space and given rise to questions even deeper and more far reaching, which embrace the entire foundation of geometry and our space conception. Let us pass in rapid review the principal events of this great movement. Wallis in the seventeenth, Saccheri, Lambert, and Legendre in the eighteenth are the first to make any noteworthy progress before the nineteenth century. The really profound investigations of Saccheri and Lambert strangely enough were entirely overlooked by later writers and have only recently come to light.

In the nineteenth century non-euclidean geometry develops along four directions which roughly follow each other chronologically. Let us consider them in order.

The naive-synthetic direction. — The methods employed are similar to those of Euclid. His axioms are assumed with the exception of the parallel axiom; the resulting geometry is what is now called hyperbolic or Lobachevsky's geometry. Its principal properties are deduced; in particular its trigonometry, which is shown to be that of a sphere with imaginary radius, as Lambert had divined. As a specific result of these investigations the long-debated question relating to the independence of the parallel axiom was finally settled. The great names in this group are Lobachevsky, Bolyai, and Gauss. The first publications of Lobachevsky are his Exposition succincte des principes de la géométrie (1829) and the Geometrische Untersuchungen (1840). Bolyai's Appendix was published in 1832. As to the extent of Gauss's investigations we can only judge from scattered remarks in private letters and his reviews of books relating to the parallel axiom. His dread of the Geschrei der Böotier, i. e., the followers of Kant, prevented him from publishing his extensive speculations.

The metric-differential direction. — This is inaugurated by three great memoirs by Riemann, Helmholtz, and Beltrami, all published in the same year, 1868.

Beltrami, making use of results of Gauss and Minding, relating to the applicability of two surfaces, shows that the hyper-

bolic geometry of a plane may be interpreted on a surface of constant negative curvature, the pseudosphere. By means of this discovery the purely logical and hypothetical system of Lobachevsky and Bolyai takes on a form as concrete and tangible as the geometry of a plane.

The work of Riemann is as original as profound. He considers space as an n-dimensional continuous numerical multiplicity which is distinguished from the infinity of other such multiplicities by certain well-defined characters. Chief of them are, 1) the quadratic differential expression which defines the length of an element of arc, and 2) a property relative to the displacements of this multiplicity about a point. There are an infinity of space multiplicities which satisfy Riemann's axioms. By extending Gauss's definition of the curvature k of a surface at a point to curvature of space at a point by considering the geodesic surfaces passing through that point, Riemann finds that all these spaces fall into three classes according as k is equal to, greater, or less than 0. For $n = 3$ and $k = 0$ we have euclidean space; when $k < 0$ we have the space found by Gauss, Lobachevsky and Bolyai; when $k > 0$ we have the space first considered in the long forgotten writings of Saccheri and Lambert, in which the right line is finite.

Helmholtz like Riemann considers space as a numerical multiplicity. To further characterize it Helmholtz makes use of the notions of rigid bodies and free mobility. His work has been revised and materially extended by Lie from the standpoint of the theory of groups.

In the present category, as also in the following one, belong important papers by Killing.

The projective direction. — We have already noticed the efforts of the synthetic school to express metric properties by means of projective relations. In this the circular points at infinity were especially serviceable. An immense step in this direction was taken by Laguerre who showed, 1853, that all angles might be expressed as anharmonic ratios with reference to these points, i. e., with reference to a certain fixed conic. The next advance is made by Cayley in his famous sixth memoir on quantics, 1859. Taking any fixed conic (or quadric, for space) which he calls the absolute, Cayley introduces two expressions depending on the anharmonic ratio with reference to the absolute. When this degenerates into the circular points at infinity, these expressions go over into the ordinary expressions

for the distance between two points and the angle between two lines. Thus all metric relations may be considered as projective relations with respect to the absolute. Cayley does not seem to be aware of the relation of his work to non-euclidean geometry. This was discovered by Klein (1871). In fact, according to the nature of the absolute various geometries are possible; among these are precisely the three already mentioned. Klein has made many important contributions to non-euclidean geometry. We mention his modification of von Staudt's definition of anharmonic ratio so as to be independent of the parallel axiom, his discovery of the two forms of Riemann's space, and finally his contributions to a class of geometries first noticed by Clifford, which are characterized by the fact that only certain of its possible displacements can affect space as a whole.

As a result of all these investigations both in the projective, as also in the metric differential direction we are led irresistibly to the same conclusion, viz.: The facts of experience can be explained by all three geometries when the constant k is taken small enough. It is therefore merely a question of convenience whether we adopt the parabolic, hyperbolic, or elliptic geometry.

The critical synthetic direction represents a return to the old synthetic methods of Euclid, Lobachevsky and Bolyai with the added feature of a refined and exacting logic. Its principal study is not of non-euclidean but of euclidean geometry. Its aim is to establish a system of axioms for our ordinary space which are complete, compatible, and irreducible. The fundamental terms, point, line, plane, between, congruent, etc., are introduced as abstract marks whose properties are determined by inter-relations in the form of axioms. Geometric intuition has no place in this order of ideas which regards geometry as a mere division of pure logic. The efforts of this school have already been crowned with eminent success, and much may be expected from it in the future. Its leaders are Peano, Veronese, Pieri, Padoa, Burali-Forti, and Levi-Civita in Italy, Hilbert in Germany, Moore in America, and Russell in England.

Closing at this point our hasty and imperfect survey of mathematics in the last century let us endeavor to sum up its main characteristics. What strikes us at once is its colossal proportions and rapid growth in nearly all directions, the great variety of its branches, the generality and complexity of its

methods, an inexhaustible creative imagination, the fearless introduction and employment of ideal elements, and an appreciation for a refined and logical development of all its parts.

We who stand on the threshold of a new century can look back on an era of unparalleled progress. Looking into the future an equally bright prospect greets our eyes; on all sides fruitful fields of research invite our labor and promise easy and rich returns. Surely this is the golden age of mathematics!

OUTER ISLAND,
September, 1904.

DE SÉGUIER'S THEORY OF ABSTRACT GROUPS.

Eléments de la Théorie des Groupes Abstraits. By J.-A. DE SÉGUIER. Paris, Gauthier-Villars, 1904. ii + 176 pp.

THE title for the complete treatise is Théorie des groupes finis. The present first volume deals with the theory as far as it demands no concrete representation. The second volume is to be entitled Compléments.

The Eléments gives a remarkably compact presentation of purely abstract group theory, including the most recent results. The attempt has been made to extend as far as possible the general theorems to infinite groups. The broader view thus gained more than compensates for the increased abstruseness. It appeals particularly to the reviewer who has given much attention to the coördination of the various branches of analytic group theory into a comprehensive theory of analytic groups in an arbitrary field. The inclusion of infinite groups, moreover, gives the author the means of a natural presentation of negative and rational numbers, Galois's imaginaries, and algebraic numbers as elements of certain groups. The author is therefore justified in giving (pages 27–51) a very compact, but practically complete, account of Galois fields (champ, corps de Galois). Relative to a first mode of composition, called addition, C_N is an additive group; relative to a second mode of composition C_N, with zero omitted, is a multiplicative group, and one may set $0x = x0 = 0$ by definition; a final postulate makes multiplication distributive with respect to addition.

The opening six pages on Cantor's assemblages establish his distinction between finite and infinite sets, but make no classification of the latter. Throughout the text the term *corps* is

used to designate a set of elements whose law of composition is associative. It is stated on page 7 that an abelian assemblage is always associative; but this is contradicted by the assemblage of all positive rational members with the law of composition $a \circ b = 1/ab$.

A group is defined by Huntington's three independent postulates (BULLETIN, volume 8, page 296); it being a theorem that the product of any two elements lies in the set. Most readers, I think, would find it more natural to have this property as a postulate, as is the case in Moore's definitions.*

The author introduces (on page 8) a *semigroup* G in connection with any subset S containing a system of generators of G. The postulates defining G are: (1) associativity; (2) for any α in S and any b in G, there is at most one solution $[x \text{ in } G]$ of $\alpha x = b$; (3) similarly for $x\alpha = b$. The author states that it follows that $ax = ax'$ requires $x = x'$ for any a in G. If so, the definition itself might better read: for any two elements α and b in G, there is at most one solution x in G of $\alpha x = b$. The conclusion was presumably reached about as follows: Express a in terms of the generators, say $a = \alpha_1\alpha_2\alpha_3$. Then $\alpha_1(\alpha_2\alpha_3 x) = \alpha_1(\alpha_2\alpha_3 x')$ would require $\alpha_2(\alpha_3 x) = \alpha_2(\alpha_3 x')$, whence $\alpha_3 x = \alpha_3 x'$, $x = x'$. However, this argument assumes that $\alpha_2\alpha_3 x$ belongs to G. But it does not follow from the postulates that the product of every two elements of G belongs to G, as stated explicitly at the bottom of page 58. Take as an example the set of any finite number of line translations to the right; the three postulates hold, but not every product occurs in the set. To include the properties desired by de Séguier, I suggest that the name semigroup be given to the very important assemblage and rule of combination defined by the following four postulates:†

(1) If a and b belong to the set, then $a \circ b$ belongs to the set.

(2) $(a \circ b) \circ c = a \circ (b \circ c)$, whenever a, b, c, $a \circ b$, $b \circ c$ $(a \circ b) \circ c$ and $a \circ (b \circ c)$ belong to the set.

(3) and [4] For every two elements a and b of the set, there exists at most one element x in the set such that $a \circ x = b$ $[x \circ a = b]$.

* *Transactions*, vol. 3 (1903), p. 485. Professor Moore has observed that his postulate $(3''_r)$ is redundant, so that his second definition becomes simpler. A simplification of his first definition may be made by changing $a'_i a$ into aa'_i in (4_l) and dropping postulate (3_l). The new definitions, which are ideally simple, will be given by Moore and the reviewer in the *Transactions* for April, 1905.

† For an assemblage satisfying postulates (1) and (2) I suggest the name algebra.

These four postulates are seen to be independent and consistent. For a semigroup, either $ax = ax'$ or $xa = xa'$ requires that $x = x'$. An example of a semigroup is the totality of the powers of an operator a of no finite period. An infinite cyclic group requires two generators a and a^{-1}.

The author's discussion of a complete set of generational relations is excellent. A point not usually noted is that arbitrarily given relations between the elements regarded as generators are never incompatible; they may of course require that certain of the generators be identical or reduce to identity. On the other hand, a multiplication table cannot be assigned arbitrarily, in view of the associative law.

The luminous account of the chief results in the abstract Galois field theory runs quite parallel to that of the reviewer's Linear Groups, the fundamental concepts being due to Galois and Moore. It is therefore not easy to understand how the author could overlook the chapters in Linear Groups which give a complete account of the theory of quadratic forms in a Galois field.* Instead, the source cited is Jordan's course of lectures at the Collège de France in 1904. I take this opportunity to state that Jordan arrived at the chief results of the theory independently.†

Passing from finite to infinite fields, a brief account is given of the most elementary properties of algebraic fields. The case of an infinite field C with a finite modulus p is disposed of in a single line (page 51) by saying "on obtient C en formant des C_{p^n} [Galois field of order p^n] où n croît indéfiniment." Unfortunately the matter is not so simple as the author thus indicates. Quite a number of years ago Professor Moore and I noticed independently the existence of the infinite field C modulo p given as the aggregate of all $GF[p^n]$, $n = 1, 2, 3, \cdots$. Replying to my query as to possible properties of an arbitrary infinite field of modulus p, he indicated the existence of fields other than C—citing the example C^* defined as the aggregate of all $GF[p^n]$, $n = m_1, m_2, \cdots$, where m_1 divides m_2, m_2 divides m_3, $\cdots$. I was later led to consider the field $C^{(p^\infty)}$ of all

* My original paper appeared in the *American Journal of Mathematics*, vol. 21 (1899), pp. 193–256.

† In a letter to me dated February 21, 1904, M. Jordan says: "J'avais préparé il y a trois ou quatre ans un travail sur les groupes linéaires à invariant quadratique et je me suis aperçu après coup que tous mes résultats se trouvaient déjà exposés dans votre bel ouvrage sur les groupes linéaires, de sorte que je n'ai eu qu'à jeter au feu tout ce que j'avais écrit."

rational functions of an arbitrary variable with coefficients in the $GF[p^n]$ and analogous infinite fields whose elements are rational functions in the $GF[p^n]$ of any number of independent variables. These types of fields have very different properties. Thus C has the very interesting property, noted by Moore, that in it every equation is completely solvable.

For two isomorphic finite groups A and B, the set A_0 of the elements of A which correspond to the identity of B form a group. For infinite groups the result is different and the author is in error. The converse statement (page 66, line 9) cannot be proved. I have constructed examples in which neither A_0 nor B_0 is a group. The correct theorem is that A_0 and B_0 are semi-groups. If either is a group the other is also, so that, for the ordinary case in which $B_0 = 1$, A_0 is a group. I will discuss this fundamental question in detail in the *Transactions*.

In the one hundred pages devoted to finite groups in the usual sense, the author has given the statement of theorems and their proofs in unusually condensed form. This is due partly to the use of Frobenius' notations throughout, partly to the use of symbols such as A_e or $A_{(e)}$ for an operator of period equal or a divisor of e respectively, and partly to the use of numerous new terms such as *central* of G for the subgroup of all the invariant operators of G. We find the new terms principal group, dicyclic, commutant, normalisant, rank, special, figure, as well as new or alternative designations in place of those in current use. It is to be hoped that a complete index will appear in the second volume.

Note I gives Jordan's work on groups of movements, now of practical interest in crystallography. Note II considers matrices, Elementartheiler, systems of linear equations and congruences.

The only new errata noted were b for b_1 on page 13, line 12; $\leqq$ for $\geqq$ on page 51, line 17. A phrase on page 62 might mislead students of Lie's theory, since "groupe continu de points $(x_1, \cdots, x_n)$" does not refer to a group of point transformations in n variables, but refers to ∞^n operators in Lie's symbolic sense.

The reader of the present volume will be impressed with the author's complete mastery of his subject and will find in it a useful compact summary of the results to date in the purely abstract part of finite group theory.

L. E. DICKSON.

SHORTER NOTICES.

An Introduction to the Modern Theory of Equations. FLORIAN CAJORI. New York, The Macmillan Company, 1904. ix + 239 pp.

A VERY clear elementary account of the ordinary theory of equations, including Gauss's 1849 proof that every algebraic equation has a root, is given in pages 1–103. Prior to the algebraic solution of the general cubic C and quartic Q, criteria of the nature of their roots are derived from the equations whose roots are the squares of the differences of every two of the roots of C and Q, respectively.

The usual elementary chapter on substitutions is followed by the elements of substitution groups, including a list of all the substitution groups of degree $\leqq 5$. It would have been instructive to the student to see the elementary proof of this enumeration for degrees $\leqq 4$.

At the end of § 110, the author omits the condition for a series of composition that P_{i+1} is a maximal normal subgroup of P_i. At the bottom of page 125 occurs simple for single.

In the author's presentation of the ideas introductory to the general Galois theory of equations, he has not maintained the usual excellence of his text, although he has avoided the gross errors of certain texts. The fundamental distinction between formal and numerical invariance is relegated to a foot-note! To the author there are just two alternatives, either the coefficients of an equation are all independent variables or are all particular numerical constants. We find the unfortunate statement at the bottom of page 125 that an equation "may represent a more general case when the coefficients are particular numbers than when they are variables." These two cases are merely the opposite extremes of the general case of Galois's theory, a point so obvious that we need not dwell on it further. The same remarks apply to page 2; furthermore, there is no reason why π or e may not enter the coefficients in Galois's theory. If the foot-note on pages 124–5 had been entirely omitted and the statement made that throughout Chapter XI the roots were regarded to be independent variables, so that "unaltered in value" means "formally unaltered," the presentation would be very satisfactory. In fact, the author has closely followed Weber in the presentation of the general Galois theory. There is a trivial discrepancy between § 121 and exercise 2. In § 159 the theorem should be given the

precision found in exercise 5. Let M be a rational function of the n roots with coefficients in the domain Ω. Let M be formally unaltered by the substitutions of a group Q and by no further substitutions on the n roots. If the conjugates of M under the symmetric group are all numerically distinct and if M is a number in Ω, the Galois group for the domain Ω is either Q or one of its subgroups.

The text furnishes interesting applications to cyclotomic equations; geometric constructions by ruler and compass, in particular to the possible divisions of the circle into equal parts; the duplication of the cube; the trisection of an angle.

L. E. DICKSON.

Annuaire Astronomique pour 1905. Hayez, Brussels, 1904. 16mo. 360 pp.

THIS handy little volume, published annually under the auspices of the Royal Belgian Observatory, would perhaps scarcely be a suitable subject for notice in these columns if the Society did not contain amongst its members many who, in addition to their mathematical duties have charge of observatories in which research work is necessarily entirely subordinate to instruction. For such it will be found useful to have on the table, along with the Nautical Almanac, its contents including besides the usual astronomical data, formulæ for finding the various terms in use, full explanations of the tables, worked examples, etc. It is published primarily to aid in the administration of the Belgian public service, to give assistance and information to all interested in astronomy, and to popularize the study of the subject. From the way in which it is written and put together it would appear to be successful in achieving these objects.

ERNEST W. BROWN.

NOTES.

THE annual meeting of the AMERICAN MATHEMATICAL SOCIETY will be held on Thursday and Friday, December 29–30. The Council will meet on Thursday morning, and the annual election of officers and other members of the Council will close on Friday morning. At the opening of the afternoon session on Thursday, the retiring President, Professor THOMAS S. FISKE, will deliver the presidential address, the subject of which will be "Mathematical progress in America."

THE Christmas meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held on Friday and Saturday, December 30 and 31, at the Northwestern University building (corner of Lake and Dearborn streets). Titles and abstracts of papers to be presented at this meeting should be in the hands of the Secretary of the Section, Professor THOMAS F. HOLGATE, 617 Hamline Street, Evanston, Ill., not later than December 8.

THE concluding (October) number of volume 5 of the *Transactions of the American Mathematical Society* contains: "On a method of dealing with the intersections of plane curves," by F. S. MACAULAY; "On certain complete systems of quaternion expressions, and on the removal of metric limitations from the calculus of quaternions," by J. M. PEIRCE; "Three particular systems of lines on a surface," by L. P. EISENHART; "On ruled surfaces whose flecnode curve intersects every generator in two coincident points," by E. J. WILCZYNSKI; "On the automorphic functions of the group (0, 3; 2, 6, 6)," by J. I. HUTCHINSON; "A theorem concerning the invariants of linear homogeneous groups, with some applications to substitution groups," by H. F. BLICHFELDT; "On the geometry whose element is the 3-point of a plane," by F. MORLEY; "Sufficient condition for a minimum with respect to one-sided variations," by G. A. BLISS; "Sur les opérations linéaires," by M. FRÉCHET; "On groups in which certain commutative operations are conjugate," by H. L. RIETZ; "On hypercomplex number systems (first paper)," by H. TABER; Notes and errata, volumes 1, 3, 4, 5; Table of contents, volume 5; Indices, volumes 1–5, by subject matter, by authors, by numbers.

THE list, recently published in the BULLETIN (pages 28–29), of American doctorates in mathematics conferred during the last academic year, with titles of dissertations, should have included: E. L. DODD, Yale, "On multiple sequences"; H. E. JORDAN, Chicago, "Group characters of various types of linear groups."

THE autumn meeting of the National academy of sciences was held at Columbia University, November 15–16. The following mathematical papers were presented: "The system of magnetic forces causing the secular variation of the earth's magnetism," by L. A. BAUER; "On topical geometry," by C.

S. PEIRCE; "The absolute value of the acceleration of gravity determined by the ring-pendulum method," by C. E. MENDENHALL; "The double suspension pendulum for measuring the acceleration of gravity," by R. S. WOODWARD.

AT a meeting held at Teachers College, Columbia University, Saturday, November 5, the New York Section of the Association of teachers of mathematics in the middle states and Maryland was organized. Professor G. LEGRAS, of the College of the City of New York, was elected chairman, and Miss CHARLOTTE SMITH of the Girls' High School of Brooklyn, secretary. The following papers were read: "The genesis of geometric concepts," by Professor C. J. KEYSER; "The teaching of algebra," by Professor J. H. TANNER.

A meeting of the Association of mathematical teachers in New England was held on November 19 at Harvard University. The program of the meeting was as follows: "An explanation of the new definitions of entrance requirements in mathematics; the reasons for the changes; the practical effects of the changes upon the teaching in the schools," the speakers being Professor H. W. TYLER, Mr. F. B. KNAPP, and Professor W. F. OSGOOD; "Some calculating machines and the slide rule," by Professor LOUIS DERR. The following officers were elected for the ensuing year: president, Mr. W. A. FRANCIS; vice-president, Professor N. F. DAVIS; secretary, Mr. G. W. EVANS.

THE Kansas association of mathematics teachers will meet at Topeka, December 28–29, in connection with the State teachers association. The opening address, by President H. B. NEWSON, will be on the subject of "High school mathematics from the standpoint of the college.

THE fifty-fourth annual meeting of the American association for the advancement of science will be held at Philadelphia, December 27–31, under the presidency of Professor W. G. FARLOW, of Harvard University. Professor A. ZIWET is vice-president and chairman of section A (mathematics and astronomy), and Professor L. G. WELD is secretary. The retiring vice-presidential address will be delivered by Mr. O. H. TITTMAN, superintendent of the coast and geodetic survey.

AMONG the new books which are in the press of J. Wiley and Son may be mentioned: An introduction to the differential

calculus, based on the method of rates, by W. W. JOHNSON; Projective geometry with applications, by A. EMCH.

THE first part of volume 1 of the French edition of the Encyclopedia of mathematics has just appeared from the presses of B. G. Teubner and Gauthier-Villars. Several other parts are in the press and arrangements have been made to publish subsequent parts as soon as possible. Members of the Deutsche Mathematiker-Vereinigung are allowed the same special price as in case of the German edition. Persons interested should communicate with Professor Dr. A. Krazer, Karlsruhe in Baden, Westendstrasse 57.

UNIVERSITY OF PARIS. — Following the affiliation with the Ecole Normale, an additional course in mathematics is announced for the present semester (compare BULLETIN, volume 10, page 520): — by Professor F. RAFFY: Applications of analysis to geometry, two hours. — A mathematical conference will be held by Professor E. GOURSAT: Applications of differential calculus to geometry: — The courses of Professors Tannery, Raffy and Borel, formerly given in the Ecole Normale, have been discontinued.

AN INTERNATIONAL prize for geometry will be awarded by the Circolo Matematico di Palermo on the occasion of the fourth international congress of mathematicians, which will be held at Rome in the year 1908. The prize, which will be styled the Guccia Medal in honour of its founder, will consist of a gold medallion and a sum of 3000 francs.

It is well known that, since the papers which were published in connection with the competition for the Steiner Prize in 1882, the theory of algebraic twisted curves has been somewhat neglected, and has not shared in the remarkable progress in geometry, which has been achieved by synthetical and algebraical methods as well as by those of the theory of functions. The result has been that neither the fundamental questions which were raised in the papers referred to, nor others which might have been propounded, have been the subjects of further investigations; and, on passing from ordinary space to space of higher dimensions, one finds a number of important problems connected with algebraic curves (especially those bearing on their classification, the study of canonical curves of given deficiency etc.) to which no attention has yet been devoted.

On the other hand but few theorems in which there is a limitation either to a real field or to a given domain of rationality are known with respect to algebraic twisted curves.

The Guccia medal is offered for "A memoir making an important advance in the theory of algebraic twisted curves." No other conditions are prescribed either as to the methods of research or as to the problems to be investigated. If none of the memoirs on the given subject is declared worthy of the prize, this may be awarded for "A memoir making an important advance in the theory of algebraic surfaces or of other algebraic manifolds."

Memoirs submitted for the competition must not have been previously published. They must be in one of the languages, Italian, French, German, or English, and type-written (except the formulæ); and three copies, inscribed with a motto, must reach the president of the Circolo Matematico di Palermo before July 1, 1907. They must be accompanied by a sealed envelope, bearing the motto and containing the name and address of the author. The memoir to which the prize is adjudged will be printed in the *Rendiconti* or in some other publication of the Circolo Matematico di Palermo. The author will receive 200 copies free of charge. If the prize is not adjudged to any of the memoirs submitted, it may be given for a published work on the theories previously mentioned, provided that this shall have appeared after the issue of the present notice and before July 1, 1907.

The prize will be awarded according to the decision of an International Commission composed of M. NOETHER, H. POINCARÉ, and C. SEGRE. The report of the Commission will be read and the prize conferred upon the successful competitor at one of the sessions of the fourth international congress of mathematicians at Rome in the year 1908.

PROFESSOR L. RAFFY has been appointed professor of differential geometry at the University of Paris.

PROFESSOR A. COTTON has been appointed professor of mechanics at the University of Grenoble.

PROFESSOR E. CARTAN has been appointed professor of the calculus at the University of Nancy.

PROFESSOR J. DRACH has been appointed professor of rational mechanics at the University of Poitiers.

MR. H. M. MACDONALD has been appointed professor of mathematics at the University of Aberdeen.

DR. O. EGGERT, of the University of Berlin, has been appointed professor of geodesy in the technical school at Danzig.

AT the Ecole Polytechnique, of Paris, Professor H. POINCARÉ has been appointed professor of general astronomy, as successor to the late Professor Callandreau. Professor LECORNU has been appointed professor of mechanics, as successor to the late Professor Sarrau.

AT the University of Lausanne, Professor H. JOLY has been promoted to a full professorship of mathematics, and Professor A. DOMMER has been promoted to a full professorship of mechanics.

PROFESSOR S. NEWCOMB has been elected a corresponding member of the Vienna academy of sciences.

PROFESSOR P. MANSION, of the University of Ghent, C. A. LAISANT, of the Ecole Polytechnique, and G. PEANO, of the University of Turin, have been elected honorary members of the Physico-mathematical society of Kasan.

THE following have been appointed examiners in the Cambridge mathematical tripos for 1905: Messrs. R. R. WEBB (St. John's), G. H. A. WILSON (Clare), J. M. DODDS (Peterhouse), E. W. BARNES (Trinity).

MR. A. YOUNG has been elected fellow in mathematics of Clare College, Cambridge.

PROFESSOR F. S. LUTHER, professor of mathematics in Trinity College, of Hartford, Conn., was installed as president of the institution October 28.

AT the University of Illinois Dr. J. STEBBINS has been promoted to an assistant professorship of astronomy. Mr. A. H. WILSON, formerly of Princeton University, has been appointed instructor in mathematics.

DR. H. B. EVANS has returned to the University of Pennsylvania after two years' absence; he has been appointed assistant professor of mathematics.

THE following appointments have been made at the University of Maine: Mr. C. P. WESTON, assistant professor of me-

chanics; Mr. H. R. WILLARD, instructor in mathematics; Mr. R. K. MORLEY, tutor in mathematics.

AT Oberlin College, Professor F. ANDEREGG has returned after a year's absence abroad; Mr. W. D. CAIRNS has been promoted to an associate professorship of mathematics; and Mr. J. R. LUCKEY has been appointed assistant in mathematics and physics.

MR. E. D. GRANT, instructor in mathematics at the Michigan College of Mines, has been promoted to an associate professorship.

DR. K. SCHMIDT has been appointed professor of mathematics and astronomy at the newly organized University of Florida, at Lake City, Florida.

THE College entrance examination board of the middle states and Maryland has reappointed the examiners in mathematics who served in 1904, namely Professor F. N. COLE, Professor T. C. ESTY, and Dr. ARTHUR SCHULTZE.

PROFESSOR H. KORTUM, of the University of Bonn, died September 24, at the age of 68 years. Professor G. LILLEY, of the University of Oregon, died June 8, aged 68 years. Professor K. EDLER VON OTT, of the German technical school at Prague, died August 23, at the age of 69 years.

RECENT catalogues of second-hand mathematical works: Librairie H. Welter, rue Bernard-Palissy 4, Paris, clearance sale of mathematical and physical works, about 1,200 titles; A. Buchholz, Ludwigstrasse 7, Munich, Antiquariats-Katalog 41, 296 titles in mathematics; Jacques Rosenthal, Karlstrasse 10, Munich, Katalog 26, 377 titles in mathematics; H. Lüneburg, Karlstrasse 4, Munich, Antiquariats-Katalog 54, Mathematik, 1,165 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOLZA (O.). Lectures on the calculus of variations. Chicago, University of Chicago Press, 1904. 8vo. 16 + 271 pp. Cloth. $4.00

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées, publiée sous les auspices des académies des sciences de Göttingue, de Leipzig, de Munich et de Vienne avec la collaboration de nombreux savants. Edition française, rédigée et publiée d'après l'édition allemande sous la direction de J. Molk. Vol. I, 1: Arithmétique, rédigé dans l'édition allemande sous la direction de F. Meyer. Premier fascicule: Principes fondamentaux de l'arithmétique; exposé, d'après l'article allemand de H. Schubert, par J. Tannery et J. Molk; Analyse combinatoire et théorie des déterminants; exposé d'après l'article allemand de E. Netto, par H. Vogt; Nombres irrationnels et notion de limite; exposé, d'après l'article allemand de A. Pringsheim, par J. Molk. Paris, Gauthier-Villars; Leipzig, Teubner; 1904. 8vo. 160 pp. Fr. 5.00

FERMAT (P.). Le plus grand quadruple (10^2, 12^2, 16^2, 22^2) carré magique. Pour la première fois édité sans fautes d'impression, par Sidoratsky. Paris, Sidoratsky, 1904. 4to. Fr. 10.00

GIBSON (G. A.). An elementary treatise on graphs. London and New York, Macmillan, 1904. 12mo. 10 + 183 pp. Cloth. $1.00

GRANVILLE (W. A.). Elements of the differential and integral calculus. With the editorial coöperation of P. F. Smith. Boston, Ginn, 1904. 8vo. 14 + 463 pp. Cloth.

LÜBECK (O.). Analytische Geometrie. Unterweisungen und Aufgaben. Strelitz, Hittenkofer, 1904. 8vo. 76 pp. (Unterrichtswerke, Methode Hittenkofer, für Selbstunterricht, Bureau- und Schulgebrauch, No. 104.) M. 3.60.

MICHEL (P.). Sulle deformate del paraboloide di rotazione. Firenze, Tipografia Galileiana, 1904. 4to. 15 pp.

MOLK (J.). See ENCYCLOPÉDIE.

MORALE (M.). Le polarità piane d'ordine *n*. Girgenti, Montes, 1904. 8vo. 16 pp.

MOSNAT (E.). Problèmes de géométrie analytique. T. 2: Géométrie à deux dimensions, à l'usage des candidats à l'Ecole polytechnique, à l'Ecole normale et l'agrégation. 2e édition. Paris, Vuibert et Nony, 1905. 8vo. 483 pp.

PFISTER (A.). Die geodätischen Linien einer Klasse von Flächen, deren Linienelement den Liouvilleschen Typus hat. (Diss.) Kiel, 1904. 8vo. 44 pp.

SCHWERING (K.). Analytische Geometrie für höhere Lehranstalten. 2te, verbesserte Auflage. Freiburg i. B., Herder, 1904. 8vo. 7 + 25 pp. M. 0.50.

SEYLER (G.). Ueber die Erhaltung der Krümmungslinien bei Orthogonal-Projektion. (Diss.) Erlangen, 1903. 8vo. 26 pp.

SIDORATSKY. See FERMAT (P.).

SMITH (P. F.). See GRANVILLE (W. A.).

TRUM (A.). Gegenseitige Verwandtschaft der Kegelschnittslinien. (Progr.) Arnau, 1904. 8vo. 21 pp.

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